Section 3

ACOUSTICS

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The Journal of the Acoustical Society of America

CONTENTS

3a. Acoustical Definitions[1]

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3a-1. General

Acceleration. The acceleration of a point is the time rate of change of the velocity of the point.

Acoustic, acoustical. The qualifying adjectives acoustic and acoustical mean containing, producing, arising from, actuated by, related to, or associated with sound. *Acoustic* is used when the term being qualified designates something that has the properties, dimensions, or physical characteristics associated with sound waves; *acoustical* is used when the term being qualified does *not* designate explicitly something which has such properties, dimensions, or physical characteristics.

EXAMPLES: Acoustic singularities manifested through acoustic impedance irregularities make possible acoustical flaw-detection methods based on acoustic flaw detection. Positive acoustical advantages can accrue from good acoustical utilization of such acoustic signals, which represent an acoustical manifestation of electricity transmitted acoustically by an acoustic medium. From the acoustical point of view, acoustic loading is an excellent method of effecting the acoustical termination of an acoustical system with an acoustic termination.

Acoustics. Acoustics is the science of sound including its production, transmission, and effects.

Anechoic Space or Chamber. An anechoic space or chamber is a bounded space in which reflected waves are sufficiently weak as to be negligible in the region of interest; more literally, echo-free space.

Antinodes (Loops). Antinodes are the points, lines, or surfaces in a standing-wave system where some characteristic of the wave field has maximum amplitude.

Note: The appropriate modifier should be used with the word "antinode" to signify the type that is intended (pressure antinode, velocity antinode, etc.).

Audio Frequency. An audio frequency is any frequency corresponding to a normally audible sound wave.

Note 1: Audio frequencies range roughly from 15 to 20,000 cycles per second.
Note 2: The word "audio" may be used as a modifier to indicate a device or system intended to operate at audio frequencies, e.g., "audio amplifier."

Band Power Level. The band power level of a sound for a specified frequency band is the acoustic power level for the acoustic power contained within the band. The width of the band and the reference power must be specified. The unit is the decibel.

Band Pressure Level. The band pressure level of a sound for a specified frequency band is the effective sound pressure level for the sound energy contained within the band. The width of the band and the reference pressure must be specified. The unit is the decibel.

[1] From American Standard Z24.1-1951, American Standards Association.

Beats. Beats are periodic variations that result from the superposition of waves having different frequencies.

Compressional Wave. A compressional wave is a wave in an elastic medium which causes an element of the medium to change its volume without undergoing rotation.

Note 1: Mathematically, a compressional wave is one whose velocity field has zero curl.
Note 2: A compressional plane wave is a longitudinal wave.

Continuous Spectrum. A continuous spectrum is the spectrum of a wave the components of which are continuously distributed over a frequency region.

Decibel. The decibel is a dimensionless unit for expressing the ratio of two values of power, the number of decibels being 10 times the logarithm to the base 10 of the power ratio.

Note: With P_1 and P_2 designating two amounts of power and n the number of decibels corresponding to their ratio,

$$n = 10 \log_{10} \frac{P_1}{P_2}$$

When the conditions are such that scalar ratios of currents or of voltages (or analogous quantities in other fields such as pressures, or particle velocities in sound) are the square roots of the corresponding power ratios, the number of decibels by which the corresponding powers differ is expressed by the following formulas:

$$n = 20 \log_{10} \frac{I_1}{I_2}$$

$$n = 20 \log_{10} \frac{V_1}{V_2}$$

where I_1/I_2 and V_1/V_2 are the given current and voltage ratios, respectively.

By extension, these relations between numbers of decibels and scalar ratios of currents or voltages are sometimes applied where these ratios are not the square roots of the corresponding power ratios; to avoid confusion, such usage should be accompanied by a specific statement of this application.

Doppler Effect. The Doppler effect is the phenomenon evidenced by the change in the observed frequency of a wave in a transmission system caused by a time rate of change in the effective length of the path of travel between the source and the point of observation.

Echo. An echo is a wave which has been reflected or otherwise returned with sufficient magnitude and delay to be perceived in some manner as a wave distinct from that directly transmitted.

Effective Particle Velocity. The effective particle velocity at a point is the root mean square of the instantaneous particle velocity (see Effective Sound Pressure for details). The unit is the meter per second (in the cgs system the unit is the centimeter per second).

Effective Sound Pressure (Root-mean-square Sound Pressure). The effective sound pressure at a point is the root-mean-square value of the instantaneous sound pressures, over a time interval at the point under consideration. In the case of periodic sound pressures, the interval must be an integral number of periods or an interval which is long compared with a period. In the case of nonperiodic sound pressures, the interval should be long enough to make the value obtained essentially independent of small changes in the length of the interval.

Note: The term "effective sound pressure" is frequently shortened to "sound pressure."

Electric Power Level, or *Sound Intensity Level.* The electric power level, or the acoustic intensity level, is a quantity expressing the ratio of two electric powers or of two sound intensities in logarithmic form. The unit is the decibel. Definitions are

$$\text{Electric power level} = 10 \log_{10} \frac{W_1}{W_2} \quad \text{db}$$

$$\text{Acoustic intensity level} = 10 \log_{10} \frac{I_1}{I_2} \quad \text{db}$$

where W_1 and W_2 are two electric powers and I_1 and I_2 are two sound intensities. Extending this thought further, we see that

$$\text{Electric power level} = 10 \log_{10} \frac{E_1^2}{R_1} \frac{R_2}{E_2^2}$$
$$= 20 \log_{10} \frac{E_1}{E_2} + 10 \log_{10} \frac{R_2}{R_1} \quad \text{db}$$

where E_1 is the voltage across the resistance R_1 in which a power W_1 is being dissipated and E_2 is the voltage across the resistance R_2 in which a power W_2 is being dissipated. Similarly,

$$\text{Acoustic intensity level} = 20 \log_{10} \frac{p_1}{p_2} + 10 \log_{10} \frac{R_{s2}}{R_{s1}} \quad \text{db}$$

where p_1 is the pressure at a point where the specific acoustic resistance (i.e., the real part of the specific acoustic impedance) is R_{s1} and p_2 is the pressure at a point where the specific acoustic resistance is R_{s2}. We note that $10 \log_{10} (W_1/W_2) = 20 \log_{10} (E_1/E_2)$ only if $R_1 = R_2$ and that $10 \log_{10} (I_1/I_2) = 20 \log_{10} (p_1/p_2)$ only if $R_{s2} = R_{s1}$.

Levels involving voltage and pressure alone are sometimes spoken of with no regard to the equalities of the electric resistances or specific acoustic resistances. This practice leads to serious confusion. It is emphasized that the manner in which the terms are used always should be clearly stated by the user in order to avoid confusion.

Flutter Echo. A flutter echo is a rapid succession of reflected pulses resulting from a single initial pulse.

Free Field. A free field is a field (wave or potential) in a homogeneous isotropic medium free from boundaries. In practice it is a field in which the effects of the boundaries are negligible over the region of interest.

Note: The actual pressure impinging on an object (e.g., electroacoustic transducer) placed in an otherwise free sound field will differ from the pressure which would exist at that point with the object removed, unless the acoustic impedance of the object matches the acoustic impedance of the medium.

Infrasonic Frequency (Subsonic Frequency). An infrasonic frequency is a frequency lying below the audio-frequency range.

Note: The word "infrasonic" may be used as a modifier to indicate a device or system intended to operate at infrasonic frequencies.

Instantaneous Particle Velocity (Particle Velocity). The instantaneous particle velocity at a point is the velocity, due to the sound wave only, of a given infinitesimal part of the medium at a given instant. It is measured over and above any motion of the medium as a whole. The unit is the meter per second (in the cgs system the unit is the centimeter per second).

Instantaneous Sound Pressure. The instantaneous sound pressure at a point is the total instantaneous pressure at that point minus the static pressure at that point. The commonly used unit is the microbar.

Intensity Level. The intensity level of a sound, in decibels, is 10 times the logarithm to the base 10 of the ratio of the intensity of this sound to a reference intensity. That is,

$$L_I = 10 \log_{10} \frac{I}{I_{\text{ref}}}$$

In the United States the reference intensity is often taken to be 10^{-16} watt/cm^2(10^{-12} watt/m^2). This reference at standard atmospheric conditions in a plane or spherical progressive wave was originally selected as corresponding approximately to the reference pressure (0.0002 microbar).

Line Spectrum. A line spectrum is the spectrum of a wave the components of which are confined to a number of discrete frequencies.

Longitudinal Wave. A longitudinal wave is a wave in which the direction of displacement at each point of the medium is normal to the wave front.

Microbar (μb). A microbar is a unit of pressure commonly used in acoustics. One microbar is equal to 0.1 newton per square meter or 1 dyne per square centimeter.

Neper. The neper is a unit used to express the scalar ratio of two currents or two voltages, the number of nepers being the natural logarithm of such a ratio.

Note 1: With I_1 and I_2 designating the scalar value of two currents, and n the number of nepers denoting their scalar ratio, then

$$n = \log_e \frac{I_1}{I_2}$$

When the conditions are such that the power ratio is the square of the corresponding current or voltage ratio, the number of nepers by which the corresponding voltages or currents differ may be expressed by the following formula:

$$n = \frac{1}{2} \log_e \frac{P_1}{P_2}$$

where P_1/P_2 is the given power ratio.

By extension, this relation between number of nepers and power ratio is sometimes applied where this ratio is not the square of the corresponding current or voltage ratio; to avoid confusion, such usage should be accompanied by a specific statement of this application.

Note 2: One neper is equal to 8.686 db.

Note 3: The neper is used in mechanics and acoustics by extending the above definition to include all scalar ratios of like quantities which are analogous to current or voltage.

Nodes. Nodes are the points, lines, or surfaces in a standing-wave system where some characteristic of the wave field has essentially zero amplitude.

Note: The appropriate modifier should be used with the word "node" to signify the type that is intended (pressure node, velocity node, etc.).

Noise. Noise is any undesired sound. By extension, noise is any unwanted disturbance within a useful frequency band, such as undesired electric waves in any transmission channel or device.

Plane Wave. A plane wave is a wave in which the wave fronts are everywhere parallel planes normal to the direction of propagation.

Power Spectrum Level. The power spectrum level of a sound at a specified frequency is the power level for the acoustic power contained in a band one cycle per second wide, centered at this specified frequency. The reference power must be specified. The unit is the decibel (see also the discussion under Pressure Spectrum Level).

Pressure Spectrum Level. The pressure spectrum level of a sound at specified frequency is the effective sound pressure level for the sound energy contained within a band one cycle per second wide, centered at this specified frequency. The reference pressure must be explicitly stated. The unit is the decibel.

Note: The concept of pressure spectrum level ordinarily has significance only for sound having a continuous distribution of energy within the frequency range under consideration. The level of a uniform band of noise with a continuous spectrum exceeds the spectrum level by

$$C_n = 10 \log_{10} (f_b - f_a) \qquad \text{db}$$

where f_b and f_a are the upper and lower frequencies of the band, respectively. The level of a uniform noise with a continuous spectrum in a band of width $f_b - f_a$ cps is therefore related to the spectrum level by the formula

$$L_n = C_n + S_n$$

where L_n = sound pressure level in decibels of the noise in the band of width $f_b - f_a$; for C_n see above, S_n = spectrum level of the noise, and n = designation number for the band being considered.

Rate of Decay. The rate of decay is the time rate at which the sound pressure level (or velocity level, or sound-energy density level) is decreasing at a given point and at a given time. The practical unit is the decibel per second.

Reverberation. Reverberation is the persistence of sound at a given point after direct reception from the source has stopped.

Note: This may be due (1) (as in the case of rooms) to repeated reflections from a small number of boundaries or to the free decay of the normal modes of vibration that were excited by the sound source, (2) (as in the case of underwater sound in the ocean) to scattering from a large number of inhomogeneities in the medium or reflection from bounding surfaces.

Shear Wave (Rotational Wave). A shear wave is a wave in an elastic medium which causes an element of the medium to change its shape without a change of volume.

Note 1: Mathematically, a shear wave is one whose velocity field has zero divergence.
Note 2: A shear plane wave in an isotropic medium is a transverse wave.

Sound-Energy Density. The sound-energy density is the sound energy in a given infinitesimal part of the gas divided by the volume of that part of the gas. The unit is the watt-second per cubic meter. (In the cgs system the unit is the erg per cubic centimeter). In many acoustic environments such as in a plane wave the sound-energy density at a point is

$$D = \frac{p^2}{\rho_0 c^2} = \frac{p^2}{\gamma P_0}$$

where γ is the ratio of specific heats for a gas and is equal to 1.4 for air and other diatomic gases. The quantity γ is dimensionless. P_0 is the barometric pressure.

Sound Field. A sound field is a region containing sound waves.

Sound Intensity (I). The sound intensity measured in a specified direction at a point is the average rate at which sound energy is transmitted through a unit area perpendicular to the specified direction at the point considered. The unit is the watt per square meter. (In the cgs system the unit is the erg per second per square centimeter.) In a plane or spherical free-progressive sound wave the intensity in the direction of propagation is

$$I = \frac{p^2}{\rho_0 c} \quad \text{watts/m}^2 \text{ or erg-sec}^{-1}/\text{cm}^2$$

Note: In the acoustical literature the intensity has often been expressed in the units of watts per square centimeter, which is equal to 10^{-7} times the number of ergs per second per square centimeter.

Sound Intensity Level. See Electric Power Level.

Sound Level. The sound level at a point in a sound field is the reading in decibels of a sound-level meter constructed and operated in accordance with the latest edition of American Standard Sound Level Meters for the Measurement of Noise and Other Sounds.[1]

The meter reading (in decibels) corresponds to a value of the sound pressure

[1] American Standard Sound Level Meters for the Measurement of Noise and Other Sounds, Z24.3-1944, American Standards Association, Inc., New York. This standard is in process of revision.

integrated over the audible frequency range with a specified frequency weighting and integration time.

Sound Power Level. The acoustic power level of a sound source in decibels is 10 times the logarithm to the base 10 of the ratio of the acoustic power radiated by the source to a reference acoustic power. That is,

$$L_W = 10 \log_{10} \frac{W}{W_{\text{ref}}} \qquad \text{db}$$

Often, W_{ref} is 10^{-13} watt. This means that a source radiating 1 acoustic watt has a power level of 130 db.

Sound Pressure Level. The pressure level of a sound, in decibels, is 20 times the logarithm to the base 10 of the ratio of the measured effective sound pressure of this sound to a reference effective sound pressure. That is,

$$L_p = 20 \log_{10} \frac{p}{p_{\text{ref}}} \qquad \text{db}$$

In the United States p_{ref} is either (1) $p_{\text{ref}} = 0.0002$ microbar (2×10^{-5} newton/m^2) or (2) $p_{\text{ref}} = 1$ microbar (0.1 newton/m^2). Reference pressure (1) has been in general use for measurements dealing with hearing and for sound-level and noise measurements in air and liquids. Reference pressure (2) has gained widespread use for calibration of transducers and some types of sound-level measurements in liquids. The two reference levels are almost exactly 74 db apart. The reference pressure must always be stated explicitly.

Spectrum. The spectrum of a wave is the distribution in frequency of the magnitudes (and sometimes phases) of the components of the wave. Spectrum also is used to signify a continuous range of frequencies, usually wide in extent, within which waves have some specified common characteristic, e.g., audio-frequency spectrum, radio-frequency spectrum, etc.

Spherical Wave. A spherical wave is a wave in which the wave fronts are concentric spheres.

Standing Waves. Standing waves are periodic waves having a fixed distribution in space which is the result of interference of progressive waves of the same frequency and kind. Such waves are characterized by the existence of nodes or partial nodes and antinodes that are fixed in space.

Static Pressure (P_0). The static pressure at a point in the medium is the pressure that would exist at that point with no sound waves present. At normal barometric pressure, P_0 equals approximately 10^5 newtons/m^2 (10^6 dynes/cm^2). This corresponds to a barometer reading of 0.751 m (29.6 in.) Hg(mercury) when the temperature of the mercury is 0°C. Standard atmospheric pressure is usually taken to be 0.760 m Hg at 0°C.

Stationary Waves. Stationary waves are standing waves in which the energy flux is zero at all points.

Note: Stationary waves can only be approximated in practice.

Strength of a Simple Sound Source. The strength of a simple sound source is the rms magnitude of the total air flow at the surface of a simple source in cubic meters per second (or cubic centimeters per second), where a simple source is taken to be a spherical source whose radius is small compared with one-sixth wavelength.

Ultrasonic Frequency (Supersonic Frequency). An ultrasonic frequency is a frequency lying above the audio-frequency range. The term is commonly applied to elastic waves propagated in gases, liquids, or solids.

Note: The word "ultrasonic" may be used as a modifier to indicate a device or system intended to operate at ultrasonic frequencies.

Velocity. The velocity of a point is the time rate of change of a position vector of that point with respect to an inertial frame.

Note: In most cases the approximation is made that axes fixed to the earth constitute an inertial frame.

Volume Velocity. The volume velocity, due to a sound wave only, is the rate of flow of the medium perpendicularly through a specified area S. That is, $U = uS$, where u is the particle velocity and U is the volume velocity. The unit is the cubic meter per second. (In the cgs system the unit is the cubic centimeter per second.)

Wave. A wave is a disturbance which is propagated in a medium in such a manner that at any point in the medium the displacement is a function of the time, while at any instant the displacement at the point is a function of the position of the point.

Any physical quantity which has the same relationship to some independent variable (usually time) that a propagated disturbance has, at a particular instant, with respect to space, may be called a wave.

Note: In this definition, displacement is used as a general term, indicating not only mechanical displacement, but also electric displacement, etc.

Wavefront. (1) The wavefront of a progressive wave in space is a continuous surface which is a locus of points having the same phase at a given instant. (2) The wavefront of a progressive surface wave is a continuous line which is a locus of points having the same phase at a given instant.

Wave Interference. Wave interference is the phenomenon which results when waves of the same or nearly the same frequency are superposed and is characterized by a spatial or temporal distribution of amplitude of some specified characteristic differing from that of the individual superposed waves.

3a-2. Sound Transmission and Propagation

Acoustic Attenuation Constant (Attenuation Constant). The acoustic attenuation constant is the real part of the acoustic propagation constant. The commonly used unit is the neper per section or per unit distance.

Note: In the case of a symmetrical structure, the imaginary parts of both the transfer constant and the acoustic propagation constant are identical, and hence either one may be called simply the attenuation constant.

Acoustic Compliance. The acoustic compliance of an enclosed volume of gas is equal to the magnitude of the ratio of the volume displacement of a piston forming one side of the volume to the pressure causing the displacement (units cm^5/dyne or m^5/newton).

Acoustic Impedance (American Standard Acoustic Impedance). The acoustic impedance at a given surface is defined as the complex ratio[1] of effective sound pressure averaged over the surface to effective volume velocity through it. The surface may be either a hypothetical surface in an acoustic medium or the moving surface of a mechanical device. The unit is newton-sec/m^5, or the mks acoustic ohm.[2] (In the cgs system the unit is dyne-sec/cm^5, or acoustic ohm.)

$$Z_A = \frac{p}{U} \qquad \text{newton-sec/m}^5 \text{ (mks acoustic ohms)}$$

Acoustic Mass (Inertance). The acoustic mass is the quantity which, when multiplied by 2π times the frequency, gives the acoustic reactance associated with the kinetic energy of the medium (units gm/cm^4 or kg/m^4).

[1] "Complex ratio" has the same meaning as the complex ratio of voltage and current in electric-circuit theory.

[2] This notation is taken from Table 12.1 of American Standard Z24.1-1951.

Acoustic Ohm. The acoustic ohm is the magnitude of an acoustic resistance, reactance, or impedance for which a sound pressure of one microbar produces a volume velocity of one cubic centimeter per second (dyne-sec/cm^5). When expressed in newton-sec/m^5, it is called the mks acoustic ohm.

Acoustic Phase Constant. The acoustic phase constant is the imaginary part of the acoustic propagation constant. The commonly used unit is the radian per section or per unit distance.

Note: In the case of a symmetrical structure, the imaginary parts of both the transfer constant and the acoustic propagation constant are identical, and have been called the "wavelength constant."

Acoustic Propagation Constant. The acoustic propagation constant of a uniform system or of a section of a system of recurrent structures is the natural logarithm of the complex ratio of the steady-state particle velocities, volume velocities, or pressures at two points separated by unit distance in the uniform system (assumed to be of infinite length), or at two successive corresponding points in the system of recurrent structures (assumed to be of infinite length). The ratio is determined by dividing the value at the point nearer the transmitting end by the corresponding value at the more remote point.

Acoustic Resistance. Acoustic resistance is the real component of the acoustic impedance. The cgs unit is the acoustic ohm. The mks unit is the specific acoustic ohm.

Acoustic, Specific Acoustic, and Mechanical Reactance. The acoustic reactance, the specific acoustic reactance, and the mechanical reactance are, respectively, the imaginary parts of the acoustic impedance, the specific acoustic impedance, and the mechanical impedance. The units are the same, respectively, as for the real, i.e., the resistive parts.

Characteristic Impedance. The characteristic impedance is the ratio of the effective sound pressure at a given point to the effective particle velocity at that point in a free, plane, progressive sound wave. It is equal to the product of the density of the medium times the speed of sound in the medium. It is analogous to the characteristic impedance of an infinitely long, dissipationless transmission line. The unit is the mks rayl, or newton-sec/m^3. (In the cgs system, the unit is the rayl, or dyne-sec/cm^3.)

Insertion Loss. The insertion loss resulting from the insertion of a transducer in a transmission system is the ratio of the power delivered to that part of the system which will follow the transducer, before insertion of the transducer, to the power delivered to that same part of the system after insertion of the transducer.

Note 1: If the input power, the output power, or both consist of more than one component, the particular components used must be specified.
Note 2: This ratio is usually expressed in decibels.

Mechanical Compliance. The mechanical compliance of a springlike device is equal to the magnitude of the ratio of the displacement of the device to the force that produced the displacement (units cm/dyne or m/newton).

Mechanical Impedance. The mechanical impedance is the complex ratio of the effective force acting on a specified area of an acoustic medium or mechanical device to the resulting effective linear velocity through or of that area, respectively. The unit is the newton-sec/m, or the mks mechanical ohm. (In the cgs system the unit is the dyne-sec/cm, or the mechanical ohm.) That is, $Z_M = f/u$ newton-sec/m (mks mechanical ohms).

Mechanical Ohm. The mechanical ohm is the magnitude of a mechanical resistance, reactance, or impedance for which a force of one dyne produces a linear velocity of one centimeter per second (dyne-sec/cm). When expressed in newton-sec/m, it is called the mks mechanical ohm.

Mechanical Resistance. Mechanical resistance is the real part of the mechanical impedance. The cgs unit is the mechanical ohm. The mks unit is the mks mechanical ohm.

Natural Frequency. A natural frequency of a body or system is a frequency of free oscillation.

Normal Mode of Vibration. A normal mode of vibration is a characteristic distribution of vibration amplitudes among the parts of the system, each part of which is vibrating freely at the same frequency. Complex free vibrations are combinations of these simple vibration forms.

Rayl.[1] The rayl is the magnitude of a specific acoustic resistance, reactance, or impedance for which a sound pressure of one microbar produces a linear velocity of one centimeter per second (dyne-sec/cm^3). When expressed in newton-sec/m^3 it is called the mks rayl.

Resonance Frequency. A resonance frequency is a frequency at which resonance exists. The commonly used unit is the cycle per second.

Note: In cases where there is a possibility of confusion, it is necessary to specify the type of resonance frequency, e.g., displacement resonance frequency or velocity resonance frequency.

Specific Acoustic Compliance. The specific acoustic compliance of a springlike device or an enclosed volume of gas is equal to the magnitude of the ratio of the displacement of the device or of a piston forming one side of the volume to the pressure that produced the displacement (units cm^3/dyne or m^3/newton).

Specific Acoustic Impedance. The specific acoustic impedance is the complex ratio of the effective sound pressure at a point of an acoustic medium or mechanical device to the effective particle velocity at that point. The unit is newton-sec/m^3, or the mks rayl. (In the cgs system the unit is dyne-sec/cm^3, or the rayl.) That is, $Z_s = p/u$ newton-sec/m^3 (mks rayls).

Specific Acoustic Mass. The specific acoustic mass is the quantity which when multiplied by 2π times the frequency gives the specific acoustic reactance associated with the kinetic energy of the medium (units gm/cm^2 or kg/m^2).

Transmission Loss. In communication, transmission loss (frequently abbreviated "loss") is a general term used to denote a decrease in power in transmission from one point to another. Transmission loss is usually expressed in decibels.

3a-3. Transmission Systems and Components

Acoustical Reciprocity Theorem. In an acoustic system comprising a fluid medium having bounding surfaces S_1, S_2, S_3, . . . , and subject to no impressed body forces, if two distributions of normal velocities v_n' and v_n'' of the bounding surfaces produce pressure fields p' and p'', respectively, throughout the region, then the surface integral of $(p''v_n' - p'v_n'')$ over all the bounding surfaces S_1, S_2, S_3, . . . , vanishes.

Note: If the region contains only one simple source, the theorem reduces to the form ascribed to Helmholtz, viz., in a region as described, a simple source at A produces the same sound pressure at another point B as would have been produced at A had the source been located at B.

Directivity Factor. (1) The directivity factor on a particular axis of a sound source is the ratio of the sound intensity at a point in the far field on the designated axis to the sound intensity that would be produced at that same point by a spherical source radiating the same total acoustic power. The frequency or the frequency band must be stated. (2) The directivity factor on a particular axis of a sound receptor (transducer, ear trumpet, etc.) is the ratio of the energy per second produced in the receptor

[1] Named in honor of Lord Rayleigh.

in response to a plane sound wave arriving along the designated axis to the energy per second that would be produced if plane sound waves having the same mean-square sound pressure were arriving simultaneously from all directions with random phase. The frequency or frequency band must be specified.

Directivity Index (Directional Gain). The directivity index of a transducer is an expression of the directivity factor in decibels, viz., 10 times the logarithm to the base 10 of the directivity factor.

Effective Acoustic Center. The effective acoustic center of an acoustic generator is the point from which the spherically divergent sound waves, observable at remote points, appear to diverge.

Effective Bandwidth. The effective bandwidth may be expressed mathematically as follows:

$$\text{Effective bandwidth} = \int_0^\infty G\,df$$

where f is the frequency in cycles per second and G is the ratio of the power transmission at the frequency f, to the transmission at the frequency of maximum transmission.

Electroacoustical Reciprocity Theorem. For an electroacoustic transducer satisfying the reciprocity principle, the quotient of the magnitude of the ratio of the open-circuit voltage at the output terminals (or the short-circuit output current) of the transducer, when used as a sound receiver, to the free-field sound pressure referred to an arbitrarily selected reference point on or near the transducer, divided by the magnitude of the ratio of the sound pressure apparent at a distance d from the reference point to the current flowing at the transducer input terminals (or the voltage applied at the input terminals), when used as a sound emitter, is a constant, called the "reciprocity constant," independent of the type or constructional details of the transducer.

Note: The reciprocity constant is given by

$$\left|\frac{M_0}{S_0}\right| = \left|\frac{M_s}{S_s}\right| = \frac{2d}{\rho f} \times 10^{-7}$$

where M_0 = free-field voltage response as a sound receiver, in open-circuit volts per microbar, referred to the arbitrary reference point on or near the transducer
M_s = free-field current response in short-circuit amperes per microbar, referred to the arbitrary reference point on or near the transducer
S_0 = sound pressure produced at a distance d cm from the arbitrary reference point in microbars per ampere of input current
S_s = sound pressure produced at a distance d cm from the arbitrary reference point in microbars per volt applied at the input terminals
f = frequency in cycles per second
ρ = density of the medium in grams per cubic centimeter
d = distance in centimeters from the arbitrary reference point on or near the transducer to the point at which the sound pressure established by the transducer when emitting is evaluated

Principal Axis. The principal axis of a transducer used for sound emission or reception is a reference direction for angular coordinates used in describing the directional characteristics of the transducer. It is usually an axis of structural symmetry, or the direction of maximum response; but if these do not coincide, the reference direction must be described explicitly.

Relative Response. The relative response is the ratio, usually expressed in decibels, of the response under some particular conditions to the response under reference conditions, which should be stated explicitly.

Response. The response of a device or system is a quantitative expression of the output as a function of the input under conditions which must be explicitly stated. The response characteristic, often presented graphically, gives the response as a function of some independent variable such as frequency or direction.

3a-4. Ultrasonics

Supersonics. Supersonics is the general subject covering phenomena associated with speed higher than the speed of sound (as in case of aircraft and projectiles traveling faster than sound).

Note: This term has been used in acoustics synonymously with "ultrasonics." Such usage is now deprecated.

Ultrasonics. Ultrasonics is the general subject of sound in the frequency range above about 15 kilocycles per second.

Ultrasonic Detector. An ultrasonic detector is a device for the detection and measurement of ultrasonic waves.

Note: Such devices may be mechanical, electrical, thermal, or optical in nature.

Ultrasonic Generator. An ultrasonic generator is a device for the production of sound waves of ultrasonic frequency.

3a-5. Hearing and Speech

Articulation (Per Cent Articulation) and *Intelligibility* (Per Cent Intelligibility). Per cent articulation or per cent intelligibility of a communication system is the percentage of the speech units spoken by a talker or talkers that is understood correctly by a listener or listeners.

The word "articulation" is customarily used when the contextual relations among the units of the speech material are thought to play an unimportant role; the word "intelligibility" is customarily used when the context is thought to play an important role in determining the listener's perception.

Note 1: It is important to specify the type of speech material and the units into which it is analyzed for the purpose of computing the percentage. The units may be fundamental speech sounds, syllables, words, sentences, etc.

Note 2: The per cent articulation or per cent intelligibility is a property of the entire communication system: talker, transmission equipment or medium, and listener. Even when attention is focused upon one component of the system (e.g., a talker, a radio receiver), the other components of the system should be specified.

Audiogram (Threshold Audiogram). An audiogram is a graph showing hearing loss, per cent hearing loss, or per cent hearing as a function of frequency.

Aural Harmonic. An aural harmonic is a harmonic generated in the auditory mechanism.

Average Speech Power. The average speech power for any given time interval is the average value of the instantaneous speech power over that interval.

Difference Limen (Differential Threshold) (Just-noticeable Difference). A difference limen is the increment in a stimulus which is just noticed in a specified fraction of the trials. The relative difference limen is the ratio of the difference limen to the absolute magnitude of the stimulus to which it is related.

Discrete Word (or *Discrete Sentence*) *Intelligibility.* Discrete word intelligibility is the per cent intelligibility obtained when the speech units considered are words (or sentences).

Electrophonic Effect. Electrophonic effect is the sensation of hearing produced when an alternating current of suitable frequency and magnitude from an external source is passed through an animal.

Hearing Loss (Deafness). The hearing loss of an ear at a specified frequency is the ratio, expressed in decibels, of the threshold of audibility for that ear to the normal threshold.[1]

[1] See also American Standard Specification for Audiometers for General Diagnostic Purposes, Z24.5-1951, or the latest revision thereof approved by the ASA.

Hearing Loss for Speech. Hearing loss for speech is the difference in decibels between the speech levels at which the average normal ear and the defective ear, respectively, reach the same intelligibility, often arbitrarily set at 50 per cent.

Instantaneous Speech Power. The instantaneous speech power is the rate at which sound energy is being radiated by a speech source at any given instant.

Loudness. Loudness is the intensive attribute of an auditory sensation, in terms of which sounds may be ordered on a scale extending from soft to loud.

Note: Loudness depends primarily upon the sound pressure of the stimulus, but it also depends upon the frequency and waveform of the stimulus.

Loudness Contours. Loudness contours are curves which show the related values of sound pressure level and frequency required to produce a given loudness sensation for the typical listener.

Loudness Level. The loudness level, in phons, of a sound is numerically equal to the sound pressure level in decibels, relative to 0.0002 μb, of a simple tone of frequency 1,000 cps which is judged by the listeners to be equivalent in loudness.

Masking. Masking is the amount by which the threshold of audibility of a sound is raised by the presence of another (masking) sound. The unit customarily used is the decibel.

Masking Audiogram. A masking audiogram is a graphical presentation of the masking due to a stated noise. This is plotted, in decibels, as a function of the frequency of the masked tone.

Mel. The mel is a unit of pitch. By definition, a simple tone of frequency, 1,000 cps, 40 db above a listener's threshold, produces a pitch of 1,000 mels. The pitch of any sound that is judged by the listener to be n times that of a 1-mel tone is n mels.

Peak Speech Power. The peak speech power is the maximum value of the instantaneous speech power within the time interval considered.

Per Cent Hearing. The per cent hearing at any given frequency is 100 minus the per cent hearing loss at that frequency.

Per Cent Hearing Loss (Per Cent Deafness). The per cent hearing loss at a given frequency is 100 times the ratio of the hearing loss in decibels to the number of decibels between the normal threshold levels of audibility and feeling.

Note 1: A weighted mean of the per cent hearing losses at specified frequencies is often used as a single measure of the loss of hearing.

Note 2: The American Medical Association has defined percentage loss of hearing for medicolegal use.[1]

Phon. The phon is the unit of loudness level. (See definition for Loudness Level.)

Pitch. Pitch is that attribute of auditory sensation in terms of which sounds may be ordered on a scale extending from low to high, such as a musical scale.

Note 1: Pitch depends primarily upon the frequency of the sound stimulus, but it also depends upon the sound pressure and waveform of the stimulus.

Note 2: The pitch of a sound may be described by the frequency of that simple tone, having a specified sound pressure or loudness level, which seems to the average normal ear to produce the same pitch.

Sone. The sone is a unit of loudness. By definition, a simple tone of frequency 1,000 cps, 40 db above a listener's threshold, produces a loudness of 1 sone. The loudness of any sound that is judged by the listener to be n times that of the 1-sone tone is n sones.

Syllable (or *Sound,* or *Vowel,* or *Consonant*) *Articulation.*[2] Syllable (or sound or

[1] See *J. Am. Med. Assoc.* **133,** 396, 397 (Feb. 8, 1947).

[2] See notes above under Articulation and Intelligibility.

vowel or consonant) articulation is the per cent articulation obtained when the speech units considered are syllables (or fundamental sounds, or vowels, or consonants).

Threshold of Audibility (Threshold of Detectability). The threshold of audibility for a specified signal is the minimum effective sound pressure of the signal that is capable of evoking an auditory sensation in a specified fraction of the trials. The characteristics of the signal, the manner in which it is presented to the listener, and the point at which the sound pressure is measured must be specified. The threshold is usually expressed in decibels relative to 0.0002 μb.

Threshold of Feeling (or Discomfort, Tickle, or Pain). The threshold of feeling (or discomfort, tickle, or pain) for a specified signal is the minimum effective sound pressure of that signal which, in a specified fraction of the trials, will stimulate the ear to a point such that there is the sensation of feeling (or discomfort, tickle, or pain). This threshold is customarily expressed in decibels relative to 0.0002 μb.

3a-6. Music

Cent. A cent is the interval between two sounds whose basic frequency ratio is the twelve-hundredth root of 2.

Note: The interval, in cents, between any two frequencies is 1,200 times the logarithm to the base 2 of the frequency ratio. Thus, 1,200 cents = 12 equally tempered semitones = 1 octave.

Complex Tone. (1) A complex tone is a sound wave produced by the combination of simple sinusoidal components of different frequencies. (2) A complex tone is a sound sensation characterized by more than one pitch.

Equally Tempered Scale. An equally tempered scale is a series of notes selected from a division of the octave (usually into 12 equal intervals, see Table 3a-1).

TABLE 3a-1. EQUALLY TEMPERED INTERVALS

Name of interval	Frequency ratio	Cents
Unison	1:1	0
Minor second or semitone	1.059463:1	100
Major second or whole tone	1.122462:1	200
Minor third	1.189207:1	300
Major third	1.259921:1	400
Perfect fourth	1.334840:1	500
Augmented fourth; diminished fifth	1.414214:1	600
Perfect fifth	1.498307:1	700
Minor sixth	1.587401:1	800
Major sixth	1.681793:1	900
Minor seventh	1.781797:1	1,000
Major seventh	1.887749:1	1,100
Octave	2:1	1,200

Fundamental Tone. (1) The funamental tone is the component in a periodic wave corresponding to the fundamental frequency. (2) The fundamental tone is the component tone of lowest pitch in a complex tone.

Harmonic. A harmonic is a partial whose frequency is an integral multiple of the fundamental frequency.

Note: The above definition is in musical terms (for the definition in physical terms, see Sound).

Harmonic Series of Sounds. A harmonic series of sounds is one in which each basic frequency in the series is an integral multiple of a fundamental frequency.

Interval. The interval between two sounds is their spacing in pitch or frequency, whichever is indicated by the context. The frequency interval is expressed by the ratio of the frequencies or by a logarithm of this ratio.

Octave. An octave is the interval between two sounds having a basic frequency ratio of 2. By extension, the octave is the interval between any two frequencies having the ratio 2:1.

Note: The interval, in octaves, between any two frequencies is the logarithm to the base 2 (or 3.322 times the logarithm to the base 10) of the frequency ratio.

Overtone. (1) An overtone is a physical component of a complex sound having a frequency higher than that of the basic frequency (see Partial below). (2) An overtone is a component of a complex tone having a pitch higher than that of the fundamental pitch.

Note: The term "overtone" has frequently been used in place of "harmonic," the nth harmonic being called the (n-1)st overtone. There is, however, ambiguity sometimes in the numbering of components of a complex sound when the word overtone is employed. Moreover, the word "tone" has many different meanings, so that it is preferable to employ terms which do not involve "tone" wherever possible.

Partial. A partial is a physical component of a sound sensation which may be distinguished as a simple tone that cannot be further analyzed by the ear and which contributes to the character of the complex sound.

Note 1: The frequency of a partial may be either higher or lower than the basic frequency and may or may not be an integral multiple or submultiple of the basic frequency (for definition of basic frequency see Basic Frequency). If the frequency is not a multiple or submultiple, the partial is inharmonic.

Note 2: When a system is maintained in steady forced vibration at a basic frequency equal to one of the frequencies of the normal modes of vibration of the system, the partials in the resulting complex tone are not necessarily identical in frequency with those of the other normal modes of vibration.

Scale. A musical scale is a series of notes (symbols, sensations, or stimuli) arranged from low to high by a specified scheme of intervals, suitable for musical purposes.

Semitone (Half Step). A semitone is the interval between two sounds whose basic frequency ratio is approximately equal to the twelfth root of 2.

Note: The interval, in equally tempered semitones, between any two frequencies, is 12 times the logarithm to the base 2 (or 39.86 times the logarithm to the base 10) of the frequency ratio.

Simple Tone (Pure Tone). (1) A simple tone is a sound wave, the instantaneous sound pressure of which is a simple sinusoidal function of the time. (2) A simple tone is a sound sensation characterized by its singleness of pitch.

Standard Pitch. The standard pitch is based on the tone A of 440 cps (see Table 3a-2).

Note 1: With this standard the frequency of middle C is 261.626 cps (see Table 3a-2).

Note 2: Musical instruments are to be capable of complying with this standard when played where the ambient temperature is 22°C (72°F).

Tone. (1) A tone is a sound wave capable of exciting an auditory sensation having pitch. (2) A tone is a sound sensation having pitch.

Whole Tone (Whole Step). A whole tone is the interval between two sounds whose basic frequency ratio is approximately equal to the sixth root of 2.

TABLE 3a-2. FREQUENCIES OF THE TONES OF THE USUAL EQUALLY TEMPERED SCALE, ARRANGED BY CORRESPONDING PIANO-KEY NUMBERS, AND BASED ON THE A OF 440 CPS

Note name	Key No.	Frequency, cps	Key No.	Frequency, cps	Key No.	Frequency, cps	Key No.	Frequency, cps	Key No.	Frequency, cps	Key No.	Frequency, cps	Key No.	Frequency, cps	Key No.	Frequency, cps
A	1	27.500	13	55.000	25	110.000	37	220.000	49	440.000	61	880.000	73	1,760.000	85	3,520.000
A#, B♭	2	29.135	14	58.270	26	116.541	38	233.082	50	466.164	62	932.328	74	1,864.655	86	3,729.310
B	3	30.868	15	61.735	27	123.471	39	246.942	51	493.883	63	987.767	75	1,975.533	87	3,951.066
C	4	32.703	16	65.406	28	130.813	40	261.626	52	523.251	64	1,046.502	76	2,093.005	88	4,186.009
C#, D♭	5	34.648	17	69.296	29	138.591	41	277.183	53	554.365	65	1,108.731	77	2,217.461		
D	6	36.708	18	73.416	30	146.832	42	293.665	54	587.330	66	1,174.659	78	2,349.318		
D#, E♭	7	38.891	19	77.782	31	155.563	43	311.127	55	622.254	67	1,244.508	79	2,489.016		
E	8	41.203	20	82.407	32	164.814	44	329.628	56	659.255	68	1,318.510	80	2,637.021		
F	9	43.654	21	87.307	33	174.614	45	349.228	57	698.456	69	1,396.913	81	2,793.826		
F#, G♭	10	46.249	22	92.499	34	184.997	46	369.994	58	739.989	70	1,479.978	82	2,959.955		
G	11	48.999	23	97.999	35	195.998	47	391.995	59	783.991	71	1,567.982	83	3,135.964		
G#, A♭	12	51.913	24	103.826	36	207.652	48	415.305	60	830.609	72	1,661.219	84	3,322.438		

3a-7. Architectural Acoustics

Anechoic Chamber. An anechoic chamber is a bounded space in which reflected waves are sufficiently weak as to be negligible in the region of interest; more literally, echo-free space.

Attenuation Constant. See Acoustic Attenuation Constant in Sec. 3a-2.

Dead Room. A dead room is a room that subjectively sounds nonreverberant. It is commonly a room having an unusually large amount of sound absorption.

Decay Constant. The decay constant is the exponential power by which sound decays after the source is stopped (units sec^{-1}).

Note: If p_0 is the effective sound pressure at $t = 0$, $p(t)$ is the effective sound pressure at time t, and the two are related by

$$p(t) = p_0 e^{-kt}$$

then k is the decay constant.

Direct Sound Wave. A direct sound wave in an enclosure is a wave emitted from a source prior to the time it has undergone its first reflection from a boundary of the enclosure.

Note: Frequently, a sound wave is said to be direct if it contains reflections that have occurred from surfaces within about 0.05 sec after the sound was first emitted.

Live Room. A live room is a room that subjectively sounds reverberant. It is commonly a room having an unusually small amount of sound absorption.

Mean Free Path. The mean free path for sound waves in an enclosure is the average distance sound travels between successive reflections in the enclosure.

Noise Reduction. In architectural acoustics, noise reduction generally is the difference between the effective sound pressure levels (in decibels) between the noise fields on opposite sides of a noise-reducing panel, with all sources of sound being on one side of the panel.

Reverberant Sound. Reverberant sound is that part of the sound in an enclosure that has undergone one or more reflections from the boundaries of the enclosure.

Reverberation Chamber. A reverberation chamber is an enclosure in which all the surfaces have been made as sound-reflective as possible. Reverberation chambers are used for certain acoustical measurements.

Room Constant. The room constant is given by the formula

$$R = \frac{S\bar{\alpha}}{1 - \bar{\alpha}}$$

where $\bar{\alpha}$ is the average sound-absorption coefficient and S is the total area of the boundaries of the room.

Sabin (Square Foot Unit of Absorption). A sabin is a measure of the sound absorption of a surface. It is the equivalent of 1 square foot of a perfectly absorptive surface.

Sound (Energy) *Absorption Coefficient.* (1) At a particular angle of wave incident, the sound-absorption coefficient is the ratio of the sound energy absorbed by the surface to the energy in the plane wave incident upon it. (2) For random wave incidence, the sound-absorption coefficient is the ratio of the sound energy absorbed by the surface to the energy incident upon it from a sound field in which sound waves are striking the surface equally from all angles of incidence. (3) The average sound-absorption coefficient for a room is the weighted average of the random-incidence absorption coefficients computed from the formula

$$\bar{\alpha} = \frac{S_1\alpha_1 + S_2\alpha_2 + S_3\alpha_3 + \cdots + S_n\alpha_n}{S_1 + S_2 + S_3 + \cdots + S_n}$$

where S_1, S_2, S_3, . . . are areas of particular surfaces in the room; α_1, α_2, α_3, . . . are the random-incidence absorption coefficients associated, respectively, with those areas; and $\bar{\alpha}$ is the average sound-absorption coefficient for the room.

Transmission Loss. In architectural acoustics, transmission loss for a wall or panel is 10 times the logarithm to the base 10 of the ratio of the sound energy incident upon the wall or panel to the sound energy transmitted through it. The transmission loss is generally measured under conditions of randomly incident sound waves. The unit is the decibel.

3b. Letter Symbols and Conversion Factors for Acoustical Quantities

LEO L. BERANEK

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T	absolute temperature, degrees Kelvin
a	absorption, energy, acoustic, total in a room
α	absorption coefficient, energy
$\bar{\alpha}$	absorption coefficient, energy, average
Y_A	acoustic admittance
C_A	acoustic compliance
G_A	acoustic conductance
x_A	acoustic excitability
y_A	acoustic immobility
Z_A	acoustic impedance (complex)
M_A	acoustic mass (inertance)
z_A	acoustic mobility
W_A, P_A	acoustic power
X_A	acoustic reactance
R_A	acoustic resistance
r_A	acoustic responsiveness
B_A	acoustic susceptance
b_A	acoustic unexcitability
g_A	acoustic unresponsiveness
Y_A	admittance, acoustic
Y_E, Y	admittance, electric
Y_M	admittance, mechanical
Y_R	admittance, rotational
Y_S	admittance, specific acoustic
A, Φ	amplitude of velocity potential
Ω	angle, solid
ϕ	angular displacement
ω	angular velocity ($2\pi f$)

f_A	antiresonant frequency
S	area (diaphragm, tube, room, or radiator)
P_0, p_0	atmospheric pressure
α	attenuation constant
$\bar{\alpha}$	average absorption coefficient, energy
C_E	capacitance, electrical
$\rho_0 c$	characteristic impedance
Q, q	charge, electric
α	coefficient of absorption
C_A	compliance, acoustic
C_S	compliance, specific acoustic
C_M	compliance, mechanical
C_R	compliance, rotational
$\xi, \eta, \zeta; \xi_x, \xi_y, \xi_z$	components of the particle displacement in the x, y, z directions
$u, v, w; u_x, u_y, u_z$	components of the particle velocity in x, y, z directions
s	condensation
G_A	conductance, acoustic
G_E, G	conductance, electric
G_M	conductance, mechanical
G_R	conductance, rotational
G_S	conductance, specific acoustic
κ	conductivity, thermal
I, i	current, electric
U	current, volume (volume per second) (volume velocity)
λ, k, δ	decay constant
D, E	density, energy
ρ	density of the medium (instantaneous)
ρ_0	density of the medium (static)
ϵ	dielectric coefficient
Δ	dilatation
D_i, DI	directivity index
R_θ	directivity ratio
ϕ	displacement, angular
ξ_x, x	displacement, linear
ξ	displacement, particle
X	displacement, volume
r	distance from source
s, x_1	distance, linear
μ	elasticity, shear
Y_E, Y	electric admittance
C_E, C	electric capacitance
Q, q	electric charge
G_E, G	electric conductance
I, i	electric current
Z_E, Z	electric impedance (complex)
P_E, W_E, P, W	electric power
X_E, X	electric reactance
R_E, R	electric resistance
ρ	electric resistivity
B_E, B	electric susceptance

E, e	electromotive force, voltage
E	energy
D, E	energy density
T, E_K	energy, kinetic
V, E_p	energy potential
x_A	excitability, acoustic
x_M	excitability, mechanical
x_R	excitability, rotational
x_S	excitability, specific acoustic
H	field strength, magnetic
m	flare coefficient in a horn
B	flux density, magnetic
Φ	flux, magnetic
f_M, F	force
f	frequency
y_A	immobility, acoustic
y_M	immobility, mechanical
y_R	immobility, rotational
y_S	immobility, specific acoustic
Z_A	impedance, acoustic (complex)
$\rho_0 c$	impedance, characteristic
Z_E, Z	impedance, electric (complex)
Z_M	impedance, mechanical (complex)
Z_R	impedance, rotational (complex)
Z_S	impedance, specific acoustic (complex)
n	index of refraction
L	inductance
M_A	inertance, acoustic mass
I	inertia, moment of
I	intensity
L_I, IL	intensity level, decibels
ν	kinematic viscosity
T, E_K	kinetic energy (inductive energy)
σ	leakage coefficient, magnetic
l	length of a vibrating string, pipe, or rod
L	level in decibels, general
x, ξ	linear displacement
s, x_1	linear distance
L, N	loudness, sones
L_N, LL	loudness level, decibels or phons
H	magnetic field strength
Φ	magnetic flux
B	magnetic flux density
σ	magnetic leakage coefficient
$\mathfrak{F}$	magnetomotive force
K, Δ	magnetostriction constant
m, M_M	mass
M_A	mass, acoustic
M_S	mass, specific acoustic

Y_M	mechanical admittance
C_M	mechanical compliance
G_M	mechanical conductance
x_M	mechanical excitability
y_M	mechanical immobility
Z_M	mechanical impedance (complex)
z_M	mechanical mobility
W_M, P_M	mechanical power
X_M	mechanical reactance
R_M	mechanical resistance
r_M	mechanical responsiveness
B_M	mechanical susceptance
b_M	mechanical unexcitability
g_M	mechanical unresponsiveness
z_A	mobility, acoustic
z_M	mobility, mechanical
z_R	mobility, rotational
z_S	mobility, specific acoustic
Y, E	modulus of elasticity
I	moment of inertia
L_{NR}, NR	noise reduction, decibels
N	number of turns
ξ	particle displacement
$\xi, \eta, \zeta; \xi_x, \xi_y, \xi_z$	particle-displacement components in the x, y, z directions
u_a	particle velocity (average)
$u, v, w; u_x, u_y, u_z$	particle-velocity components in the x, y, z directions
u_i	particle velocity (instantaneous)
u_m	particle velocity (maximum)[1]
u_p	particle velocity (peak)[1]
u	particle velocity (rms)
P	perimeter
T	period $T = 1/f$
θ, ϕ, ψ	phase angle
β	phase constant
f_{ij}, g_{ij}, d_{ij}	piezoelectric constants
σ	Poisson's ratio
Y, P	porosity (of an acoustical material)
V, E_p	potential energy (capacitive energy)
ϕ	potential, velocity
A, Φ	potential, velocity, amplitude
W, P	power
W_A, P_A	power, acoustic
W_E, P_E, P, E	power, electric
W_M, P_M	power, mechanical
W_R, P_R	power, rotational
P_0, p_0	pressure, atmospheric
p_a	pressure, sound (average)
p	pressure, sound (rms)
p_i	pressure, sound (instantaneous)

[1] For definitions of "peak" and "maximum" see American Standard Acoustical Terminology (ASA Z24.1-1951).

p_m	pressure, sound (maximum)[1]
p_p	pressure, sound (peak)[1]
γ, Γ	propagation constant
a	radius of a diaphragm, tube, or radiator
Q	ratio of mass (or inductive) reactance to resistance
γ	ratio of specific heats
X_A	reactance, acoustic
X_E, X	reactance, electric
X_M	reactance, mechanical
X_R	reactance, rotational
X_S	reactance, specific acoustic
n	refraction, index of
τ	relaxation time
$\mathcal{R}$	reluctance
R_A	resistance, acoustic
R_E, R	resistance, electric
R_M	resistance, mechanical
R_R	resistance, rotational
R_S	resistance, specific acoustic
ρ	resistivity, electrical
r_A	responsiveness, acoustic
r_M	responsiveness, mechanical
r_R	responsiveness, rotational
r_S	responsiveness, specific acoustic
T, t_{60}	reverberation time
R	room constant $\bar{\alpha}S/(1 - \bar{\alpha})$
Y_R	rotational admittance
C_R	rotational compliance
G_R	rotational conductance
x_R	rotational excitability
y_R	rotational immobility
Z_R	rotational impedance (complex)
z_R	rotational mobility
W_R, P_R	rotational power
X_R	rotational reactance
R_R	rotational resistance
r_R	rotational responsiveness
B_R	rotational susceptance
b_R	rotational unexcitability
g_R	rotational unresponsiveness
L_S, SL	sensation level, decibels
μ	shear elasticity
A, U_0	simple source strength
Ω	solid angle
L_W, PWL	sound power level, decibels
p_a	sound pressure (average)
p_i	sound pressure (instantaneous)
p_M	sound pressure (maximum)[1]
p_p	sound pressure (peak)[1]

[1] For definitions of "peak" and "maximum" see American Standard Acoustical Terminology (ASA Z24.1-1951).

p	sound pressure (rms)
L_P, SPL	sound pressure level, decibels
A, U_0	source, simple, strength of
r	source, distance from
Y_S	specific acoustic admittance
C_S	specific acoustic compliance
G_S	specific acoustic conductance
x_S	specific acoustic excitability
y_S	specific acoustic immobility
M_S	specific acoustic mass
z_S	specific acoustic mobility
X_S	specific acoustic reactance
R_S	specific acoustic resistance
r_S	specific acoustic responsiveness
B_S	specific acoustic susceptance
b_S	specific acoustic unexcitability
g_S	specific acoustic unresponsiveness
γ	specific heats, ratio of
c	speed of sound
s, S	stiffness
A, U_0	strength of a simple source
B_A	susceptance, acoustic
B_E, B	susceptance, electric
B_M	susceptance, mechanical
B_R	susceptance, rotational
B_S	susceptance, specific acoustic
G_x	system-rating constant
T	temperature, absolute, degrees Kelvin
F	tension (force) in a membrane or string
κ	thermal conductivity
t	thickness
t	time
τ	time, relaxation
T, t_{60}	time, reverberation
f_R, T	torque
a	total acoustical (energy) absorption in a room
τ	transmission coefficient, energy, barriers
L_T, TL	transmission loss of building structures, decibels
N	turns, number of
b_A	unexcitability, acoustic
b_M	unexcitability, mechanical
b_R	unexcitability, rotational
b_S	unexcitability, specific acoustic
g_A	unresponsiveness, acoustic
g_M	unresponsiveness, mechanical
g_R	unresponsiveness, rotational
g_S	unresponsiveness, specific acoustic
u	velocity
ω	velocity, angular ($2\pi f$)
c	velocity of sound
u_a	velocity, particle (average)

u_i	velocity, particle (instantaneous)
u_m	velocity, particle (maximum)[1]
u_p	velocity, particle (peak)[1]
u	velocity, particle (rms)
ϕ	velocity potential
A, Φ	velocity potential amplitude
U	velocity, volume
η	viscosity, dissipative or frictional
ν	viscosity, kinematic
E, e	voltage, electromotive force
V	volume
U	volume current; volume velocity
X	volume displacement
U	volume velocity; volume current
λ	wavelength
k	wave number, $\frac{\omega}{c} = \frac{2\pi f}{c} = \frac{2\pi}{\lambda}$
w	width
Y, E	Young's modulus

TABLE 3b-1. CONVERSION FACTORS FOR ACOUSTICAL QUANTITIES

Multiply the number of	By	To obtain the number of	Conversely multiply by
Acoustic ohms	10^5	Mks acoustic ohms	10^{-5}
Atmospheres	406.80	Inches of water at 4°C	2.458×10^{-3}
Centimeters	10^{-2}	Meters	10^2
Cubic centimeters	10^{-6}	Cubic meters	10^6
Dynes	10^{-5}	Newtons	10^5
Dynes/cm²	10^{-1}	Newtons per square meter	10
Ergs	10^{-7}	Joules	10^7
Ergs per second	10^{-7}	Watts	10^7
Ergs per second/cm²	10^{-3}	Watts per square meter	10^3
Gauss	10^{-4}	Webers per square meter	10^4
Kilograms	10^3	Grams	10^{-3}
Mechanical ohms	10^{-3}	Mks mechanical ohms	10^3
Meters	10^2	Centimeters	10^{-2}
Microbars	10^{-1}	Newtons per square meter	10
Newtons	10^5	Dynes	10^{-5}
Newtons per square meter	10	Dynes per square centimeter	10^{-1}
Pounds per square foot	0.4882	Grams per square centimeter	2.0481
Rayls	10	Mks rayls	10^{-1}
Watts per square meter	10^{-4}	Watts per square centimeter	10^4
Webers per square centimeter	10^4	Gauss	10^{-4}

[1] For definitions of "peak" and "maximum" see American Standard Acoustical Terminology (ASA Z24.1-1951).

3c. Propagation of Sound in Fluids

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3c-1. Glossary of Symbols[1]

a, a_1; a_i	material coordinate (31); surface element (12)
A; $\mathbf{A}_1$	surface (12), attenuation per wavelength (76), Avogadro's number (95); first order vector potential
B	coefficient relating $\nabla\rho$ and ∇p (58)
c, c_0; c^0, c^∞	speed of sound, reference speed (25); low- and high-frequency limit speeds (84)
c'	speed of thermal wave (78*b*)
C_p, C_v	specific heats at constant pressure, constant volume (14)
d_{ij}	rate of deformation tensor (9)
D	material differential operator (2)
E, F, G, H	algebraic abbreviations (74)
E, E_k, E_I; E_{diss}	energy densities per unit mass (60), (12); degraded component of internal energy (66)
f, $\mathbf{f}_v$, $f(\ \ )$, $f(h)$	frequency, sum of viscosity terms (62), "function of" (45), special tabulated function (75)
Δf_c	critical bandwidth (98)
F_i, $\mathbf{F}$	vector body force per unit mass (6)
$g(h)$	tabulated function (75)
h	material mass coordinate (37), argument of tabulated function (75), Planck's constant (89)
i, j, k	coordinate indexes (1)
$\mathbf{I}$	average sound-energy-flux density = sound intensity (64)
j	designation of imaginary axis, $[e^{+j\omega t}]$ (69)
$\mathbf{J}$	sound-energy flux vector (54)
k, k_0	phase constant $= \omega/c = 2\pi/\lambda$, Boltzmann's constant (89), $k_0 = \omega/c_0 = 2\pi/\lambda_0$ (47)
K; K_s, K_0, K_T	elastic modulus $= -V(DP/DV)$ (25), material constant $= c^0/c^\infty$ (84); isentropic modulus, reference modulus, isothermal modulus
L	mean free path (86), a sum of linear dimensions (90)
M	peak particle-velocity Mach number $= \omega\xi_0/c_0$ (49), molecular weight (95)
n_v	total number of molecules per unit volume (95)
N	number of modes of vibration (90)
$0(\ \ )$	additive terms of indicated order of magnitude (76)

[1] Numbers indicate equation number in or near which quantity is defined.

p; p_1, p_2	incremental, or sound, pressure; first- and second-order sound pressures (25)
P, P_0; P_m, P_{th}	total pressure (7), equilibrium or reference pressure (25); mean pressure (7), thermodynamic pressure (14)
P_1, P_2; $\mathcal{P}$	rms fundamental and second-harmonic pressure (49*a*); Prandtl number (72)
$\mathbf{q}$, q_i; $\mathfrak{q}$	heat flux vector (12); Stokes radiation coefficient (21*b*)
q; q^E, q^L	exemplar of state or condition variable (39); superscript indicates function of spatial (E) variables, or material (L) variables (32*b*)
$\mathbf{R}$, R; R_1, R_2	vorticity $= \frac{1}{2}\nabla \times \mathbf{u}$ (11*d*), real part of complex impedance; first- and second-order components of vorticity (57)
s, s_1	specific entropy per unit mass (14), first-order condensation $= \rho_1/\rho_0$ (59)
S; S'; S_{irr}	Stokes number $= \omega\eta/\rho_0 c_0^2$ (72), total interior surface (90); frequency number for radiation $= \omega/\mathfrak{q}$ (72); entropy generated irreversibly (15*a*)
t; t_{ij}	time (2); stress tensor (6)
T	absolute temperature (12)
$\mathbf{u}$, u_1; u_1, u_2, u_3	particle velocity (1); velocity components
$\mathbf{u}_1$, $\mathbf{u}_2$	first- and second-order components of particle velocity (25)
v; $\bar{v}$	specific volume $= \rho^{-1}$ (1); mean molecular velocity (86)
$\mathcal{V}$	viscosity number $= 2 + \eta'/\eta$ (10)
V; V_{ij}	volume (1); residual stress tensor (7)
x_1, x_2, x_3	cartesian coordinates (1)
X; X'	frequency number $= \omega\eta\mathcal{V}/\rho_0 c_0^2$ (72), specific acoustic reactance (69); frequency number for relaxation (84)
Y	thermoviscous number $= \kappa/\eta\mathcal{V}C_p$ (72)
z, Z	specific acoustic impedance ratio (87), and impedance (69)
α; α_K, α_C	attenuation constant (69); "Kirchhoff" and "classical" attenuation (79*a*,*b*)
β; β_{noise}	coefficient of thermal expansion $= \rho(\partial v/\partial T)_P$ (22); spectrum level $= 10 \log_{10} d(p^2/p_0^2)/df$ (98)
γ	ratio of specific heats $= C_p/C_v$ (14)
δ; δ_{ij}; Δ	finite increment (32); Kronecker delta (7); dilatation rate $= \nabla \cdot \mathbf{u}$ (4)
ϵ	specific internal energy per unit mass (13)
η, η', η_B	coefficient of shear viscosity (10), "second" or dilatational viscosity (10), bulk viscosity (10)
θ_1, θ_2	first- and second-order variational components of temperature (25)
κ	thermal conductivity (21*a*)
λ; λ_0	wavelength $= c/f$ (47); $\lambda_0 = c_0/f$
ν, ν', ν_B	kinematic viscosity coefficients (10) $= \eta/\rho$, etc.
ξ; ξ_t	displacement of particle from equilibrium (31); partial derivative with respect to subscript variable (41*b*)
ρ, ρ_0; ρ_1, ρ_2	densities: total, equilibrium; first- and second-order variational components
τ_r, τ_v, τ_k	relaxation times (83, 85)
φ_2; ϕ_η, ϕ_k	scalar velocity potential (55); viscous and thermal dissipation functions (16, 18)
χ	complex propagation constant $= \alpha + jk$ (69)
ψ	functional relation (71)
ω; ω_r, ω_v, ω_k	angular frequency $= 2\pi f$; relaxation angular frequencies (84)
∇, $\nabla\cdot$, $\nabla\times$	gradient, divergence, and curl operators
$\langle\ \rangle$	time average

3c-2. The Motion of Viscous Fluids. The motions of a fluid medium that comprise sound waves are governed by equations that include (1) a continuity equation expressing the conservation of mass, (2) a force equation expressing the conservation of momentum, (3) a heat-exchange equation expressing the conservation of energy, and (4) one or more defining equations expressing the constitutive relations that characterize the medium and its response to thermal or mechanical stress. These equations will first be presented in their complete exact form in order to provide a rigorous point of departure for the approximations that must ultimately be made in formulating the linearized, or small-signal, acoustic equations.

The transformation properties of these equations can be indicated by writing them in either vectorial or tensorial form, and both forms will be exhibited in order to facilitate contacts with the rich literature dealing with the motion of fluids.[1]

Cartesian spatial coordinates will be designated x_1, x_2, x_3, and the vector velocity of a material particle will be identified as $\mathbf{u}$ with components u_1, u_2, u_3. These will also be written as x_i and u_i, where it is implied that the subscript i, j, or k takes on successively the values 1, 2, 3. The term "material particle" denotes a finite mass element of the medium small enough for the values assumed by the state variables at every interior point of the particle not to differ significantly from the values they have at the interior reference point whose coordinates "locate" the particle.

Equation of Continuity. The conservation of mass requires that $\rho V = \rho_0 V_0$, where ρ_0 and V_0 are initial and ρ and V are subsequent values assumed by the density and volume of a particular material element of the medium. It follows that

$$\rho DV + VD\rho = 0 \qquad \frac{DV}{V} = -\frac{D\rho}{\rho} \tag{3c-1}$$

If $\rho_0 V_0$ is set equal to 1, V_0 becomes the *specific volume*, $v \equiv 1/\rho$; whence the relation between the total logarithmic time derivatives of v and ρ is

$$\frac{1}{v}\frac{Dv}{Dt} = -\frac{1}{\rho}\frac{D\rho}{Dt} = \frac{D \log v}{Dt} = -\frac{D \log \rho}{Dt} \tag{3c-2}$$

where $D(\ \ )/Dt$ denotes the "material" derivative, i.e., one that follows the motion of a material "particle" of the medium relative to a fixed spatial coordinate system, and is defined by

$$\frac{D(\ \ )}{Dt} \equiv \frac{\partial(\ \ )}{\partial t} + \mathbf{u} \cdot \mathbf{grad}\ (\ \ ) \equiv \frac{\partial(\ \ )}{\partial t} + u_i \frac{\partial(\ \ )}{\partial x_i} \tag{3c-3}$$

Analysis of the rate of deformation of a volume element yields the kinematical relation

$$\frac{1}{v}\frac{Dv}{Dt} = \operatorname{div} \mathbf{u} \equiv \Delta = \frac{\partial u_i}{\partial x_i} \tag{3c-4}$$

where Δ is the *dilatation rate.* Note that in the last terms of (3c-3) and (3c-4) summation is implied over all the allowable values of the subscript index. Equations (3c-2), (3c-3), and (3c-4) can be combined to yield the following equivalent forms of Euler's *continuity equation:*

[1] A definitive restatement of the classical-continuum point of view, with critical comments on more than 800 bibliographical references, has been given by C. Truesdell, The Mechanical Foundations of Elasticity and Fluid Dynamics, *J. Rational Mechanics and Analysis* **1**, 125–300 (January and April, 1952), and Corrections and Additions . . . , *J. Rational Mechanics and Analysis* **2**, 593–616 (July, 1953). See also Lamb, "Hydrodynamics," 6th ed., Dover Publications, New York, 1945; Rayleigh, "Theory of Sound," 2d ed. rev., Dover Publications, New York, 1945; and L. Howarth, ed., "Modern Developments in Fluid Dynamics, vol. I, chap. III, Oxford University Press, New York, 1953.

$$\frac{D\rho}{Dt} + \rho\frac{\partial u_i}{\partial x_i} = \frac{\partial \rho}{\partial t} + u_i\frac{\partial \rho}{\partial x_i} + \rho\frac{\partial u_i}{\partial x_i} = \frac{D\rho}{Dt} + \rho \operatorname{div} \mathbf{u} = 0$$
$$= \frac{1}{\rho}\frac{D\rho}{Dt} + \Delta = \frac{\partial \rho}{\partial t} + \mathbf{u}\cdot\mathbf{grad}\,\rho + \rho\operatorname{div}\mathbf{u}$$
$$= \frac{\partial \rho}{\partial t} + \mathbf{u}\cdot\nabla\rho + \rho\nabla\cdot\mathbf{u} = \frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) \qquad (3c\text{-}5)$$

In the last line of (3c-5), the Gibbs-Hamilton notation has been used for the differential vector operators, $\nabla \equiv$ **grad**; $\nabla\cdot \equiv$ div; $\nabla\times \equiv$ **curl.**

Force Equation. The linear-momentum principle can be stated in terms of Cauchy's first law of motion,

$$\rho\frac{Du_i}{Dt} = \rho F_i + \frac{\partial t_{ij}}{\partial x_j} \qquad (3c\text{-}6)$$

where the vector F_i is an extraneous body force per unit mass, and where t_{ij} is a second-rank *stress tensor* that represents the net mechanical action of contiguous material on a volume element of the medium due to the actual forces of material continuity. For an isotropic medium in which the stress is a linear function of the rate of deformation, as here assumed, the stress tensor can be resolved arbitrarily as the sum of a scalar, or hydrostatic, pressure function P and a residual stress tensor V_{ij} defined by

$$t_{ij} = -P\delta_{ij} + V_{ij} \qquad t_{ij} = t_{ji} \qquad (3c\text{-}7)$$

where δ_{ij} is the Kronecker delta which equals unity if $i = j$, but is zero otherwise. Unless V_{ii} vanishes, P is *not* identical with the mean pressure, $P_m = -\frac{1}{3}t_{ii}$. The resolution given by (3c-7) is both unique and useful, however, if P is made equal to the thermodynamic pressure P_{th} defined below. Then the residual stress tensor is given, to a first approximation, by the linear terms of an expansion in powers of the viscosity coefficients;

$$V_{ij} = \eta' d_{kk}\delta_{ij} + 2\eta d_{ij} \qquad V_{ij} = V_{ji} \qquad (3c\text{-}8)$$

in which d_{ij} is the *rate of deformation* tensor defined by

$$d_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \qquad (3c\text{-}9)$$

and where η is the "first," or conventional shear, viscosity coefficient. In accordance with current proposals for standardization, η' replaces λ, the symbol used by Stokes, Rayleigh, Lamb, et al., to designate the "second," or dilatational, viscosity coefficient. The term "bulk" viscosity is reserved for $(\lambda + \frac{2}{3}\mu) \to (\eta' + \frac{2}{3}\eta)$, the linear combination of coefficients that vanishes when the *Stokes relation* holds. Thus, $\eta \equiv$ first, or shear, viscosity; $\eta' \equiv$ second, or dilatational, viscosity; $\eta_B \equiv \eta' + \frac{2}{3}\eta =$ bulk viscosity; $\nu \equiv \eta/\rho$; $\nu' \equiv \eta'/\rho$; $\nu_B \equiv \eta_B/\rho$ (kinematic viscosities);

$$(\lambda + 2\mu) \to \eta' + 2\eta = \eta_B + \frac{4}{3}\eta = \eta\left(\frac{4}{3} + \frac{\eta_B}{\eta}\right) = \eta\mathcal{V} \qquad (3c\text{-}10)$$
$$\mathcal{V} \equiv \frac{4}{3} + \frac{\eta_B}{\eta} = 2 + \frac{\eta'}{\eta} \equiv \text{viscosity number}$$

Putting (3c-7), (3c-8), (3c-9) into (3c-6) yields the vector *force equation* in the following equivalent forms:

$$\rho\frac{\partial u_i}{\partial t} + \rho u_j\frac{\partial u_i}{\partial x_j} = \rho F_i - \frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}(\eta' d_{kk}\delta_{ij} + 2\eta d_{ij})$$
$$= \rho F_i - \frac{\partial P}{\partial x_i} + \eta'\frac{\partial^2 u_k}{\partial x_i \partial x_k} + \eta\frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$
$$+ \frac{\partial u_k}{\partial x_k}\frac{\partial \eta'}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\frac{\partial \eta}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\frac{\partial \eta}{\partial x_j} \qquad (3c\text{-}11a)$$

$$\rho \frac{D\mathbf{u}}{Dt} = \rho\mathbf{F} - \mathbf{grad}\, P + (\eta' + \eta)\, \mathbf{grad}\, (\mathrm{div}\, \mathbf{u}) + \eta\nabla^2(\mathbf{u})$$
$$+ (\mathrm{div}\, \mathbf{u})\, \mathbf{grad}\, \eta' + 2\, (\mathbf{grad}\, \eta \cdot \mathbf{grad})\, \mathbf{u} + \mathbf{grad}\, \eta \times \mathbf{curl}\, \mathbf{u} \quad \text{(3c-11}b\text{)}$$

$$\rho \frac{\partial \mathbf{u}}{\partial t} = \rho\mathbf{F} - \rho(\mathbf{u} \cdot \nabla)\mathbf{u} - \nabla P + (\eta' + 2\eta)\nabla(\nabla \cdot \mathbf{u}) - \eta\nabla \times (\nabla \times \mathbf{u})$$
$$+ (\nabla \cdot \mathbf{u})\nabla\eta' + 2(\nabla\eta \cdot \nabla)\mathbf{u} + \nabla\eta \times (\nabla \times \mathbf{u}) \quad \text{(3c-11}c\text{)}$$

The vorticity, defined by $\mathbf{R} = \frac{1}{2}\, \mathbf{curl}\, \mathbf{u} = \frac{1}{2}(\nabla \times \mathbf{u})$, and the dilatation rate, $\Delta \equiv \nabla \cdot \mathbf{u}$, can be introduced as useful abbreviations. A somewhat more symmetrical expression in terms of the mass transport velocity $\rho\mathbf{u}$ is obtained if the last form of the continuity equation (3c-5) is multiplied by $\mathbf{u}$ and added to (3c-11c), giving

$$\frac{\partial(\rho\mathbf{u})}{\partial t} + \mathbf{u}(\nabla \cdot \rho\mathbf{u}) + (\rho\mathbf{u} \cdot \nabla)\mathbf{u} = \rho\mathbf{F} - \nabla P + \eta\mathcal{V}\nabla\Delta - 2\eta\nabla \times \mathbf{R} + \Delta\nabla\eta'$$
$$+ 2(\nabla\eta \cdot \nabla)\mathbf{u} + 2\nabla\eta \times \mathbf{R} \quad \text{(3c-11}d\text{)}$$

These equations reduce to the so-called *Navier-Stokes equations* when it is assumed that η and η' are constant ($\nabla\eta = \nabla\eta' = 0$) and that the Stokes relation holds ($\eta_B = 0$, $\mathcal{V} = \frac{4}{3}$); and still further simplification follows if the motion is assumed irrotational so that $\mathbf{R} = 0$. If the viscosity coefficients are to be regarded as functions of one or more of the state variables, however, the gradients of the η's must be retained so that the implicit functional dependence can be introduced by writing, for example, $\nabla\eta = (\partial\eta/\partial T)\nabla T + \cdots$.

Energy Relations and Equations of State. The conservation of energy requires that the following power equation be satisfied:

$$\frac{D(E_k + E_I)}{Dt} = \int_V \rho F_i u_i\, dV + \int_A t_{ij} u_j\, da_i - \int_A q_i\, da_i \quad \text{(3c-12)}$$

where E_k is the kinetic energy associated with the material velocity, E_I is the total internal energy, V is a volume bounded by the surface A, da_i is the projection of a surface element of A on the plane normal to the $+x_i$ axis, F_i is the extraneous body force (per unit mass), and q_i is the total heat flux vector (mechanical units). After the surface integrals are converted to volume integrals by using the divergence theorem, and with the help of (3c-6), this equation reduces to the Fourier–Kirchhoff–C. Neumann[1] energy equation,

$$\rho \frac{D\epsilon}{Dt} = t_{ij} d_{ij} - \frac{\partial q_i}{\partial x_i} \quad \text{(3c-13)}$$

where ϵ is the local value of the specific internal energy (per unit mass) defined through $E_I = \int_V \rho\epsilon\, dV$. It is now postulated that the state of the fluid is completely specified by ϵ and two other local state variables, which can be taken as the specific entropy s (per unit mass) and the specific volume $v = \rho^{-1}$, in terms of which the thermodynamic pressure and temperature, and the specific heats can be defined by

$$\epsilon = \epsilon(s,v) \qquad P_{\mathrm{th}} \equiv -\left(\frac{\partial\epsilon}{\partial v}\right)_s \qquad T \equiv \left(\frac{\partial\epsilon}{\partial s}\right)_v$$
$$C_p \equiv T\left(\frac{\partial s}{\partial T}\right)_p \qquad C_v \equiv T\left(\frac{\partial s}{\partial T}\right)_v \qquad \gamma \equiv \frac{C_p}{C_v} \quad \text{(3c-14)}$$

The second law of thermodynamics can be introduced in the form of an equality, which replaces the classical Clausius-Duhem inequality, through the expedient of accounting explicitly for the creation of entropy S_{irr} (per unit volume) by irreversible

[1] See footnote, p. 3–27.

dissipative processes;[1] thus

$$\frac{D}{Dt}\int_V \rho s\,dV = -\int_A \frac{q_i}{T}\,da_i + \int_V \frac{DS_{\text{irr}}}{Dt}\,dV \tag{3c-15a}$$

This relation states that the increase of entropy in a material element is accounted for by the influx of heat and by the irreversible production of entropy within the element. The left-hand side of (3c-15*a*) can also be written, with the help of the continuity relation, as $\int_V \rho(Ds/Dt)\,dV$. Then, after converting the surface integral to a volume integral, the second law can be given in differential form as

$$\begin{aligned}\rho\frac{Ds}{Dt} &= -\frac{\partial}{\partial x_i}\frac{q_i}{T} + \frac{DS_{\text{irr}}}{Dt}\\ &= -\frac{1}{T}\frac{\partial q_i}{\partial x_i} + \frac{q_i}{T^2}\frac{\partial T}{\partial x_i} + \frac{DS_{\text{irr}}}{Dt}\end{aligned} \tag{3c-15b}$$

A thermal-dissipation function ϕ_k can be defined by

$$\phi_\kappa = -\frac{q_i}{T}\frac{\partial T}{\partial x_i} \tag{3c-16}$$

whereupon multiplying (3c-15*b*) by T yields the second-law equality in the form

$$\rho T\frac{Ds}{Dt} = -\frac{\partial q_i}{\partial x_i} - \phi_\kappa + T\frac{DS_{\text{irr}}}{Dt} \tag{3c-15c}$$

Taking the material derivative of the basic equation of state (3c-14_1) (where the subscript added to an equation number indicates the serial number of the equality sign to which reference is made when several relations are grouped under one marginal identification number), introducing the definitions for P_{th} and T, multiplying by ρ, and using (3c-4), gives

$$\rho T\frac{Ds}{Dt} = \rho\frac{D\epsilon}{Dt} + P_{\text{th}}\Delta \tag{3c-17}$$

The energy equation (3c-13) can be recast, using (3c-7) and (3c-9), in the form

$$\rho\frac{D\epsilon}{Dt} + P\Delta + \frac{\partial q_i}{\partial x_i} = V_{ij}d_{ij} = \phi_\eta \tag{3c-18}$$

in which $V_{ij}d_{ij}$, the dissipative component of the stress power $t_{ij}d_{ij}$, is defined as the viscous dissipation function ϕ_η. The usefulness of specifying the arbitrary scalar in (3c-7) as the thermodynamic pressure, so that $P = P_{\text{th}}$, becomes apparent when $\rho\,D\epsilon/Dt$ is eliminated between (3c-18) and (3c-17), giving

$$\begin{aligned}\rho T\frac{Ds}{Dt} &= (P_{\text{th}} - P)\Delta + \phi_\eta - \frac{\partial q_i}{\partial x_i}\\ &= \phi_\eta - \frac{\partial q_i}{\partial x_i}\end{aligned} \tag{3c-19}$$

The viscous dissipation function (dissipated energy per unit volume) is thus seen to account for either an efflux of heat or an increase of entropy. Subtracting (3c-19) from (3c-15*c*) then allows the rate of irreversible production of entropy to be evaluated directly in terms of the two dissipation functions,

$$T\frac{DS_{\text{irr}}}{Dt} = \phi_\eta + \phi_\kappa \tag{3c-20}$$

The total heat-flux vector q_i, whose divergence is the energy transferred *away* from the volume element, must account for energy transport by either conduction or radi-

[1] Tolman and Fine, *Revs. Modern Phys.* **20,** 51–77 (1948).

ation. The part due to conduction is given by the Fourier relation, which serves also to define the heat conductivity κ;

$$(q_i)_{\text{cond}} = -\kappa \frac{\partial T}{\partial x_i}$$

$$\frac{\partial (q_i)_{\text{cond}}}{\partial x_i} = -\frac{\partial(\kappa \partial T/\partial x_i)}{\partial x_i} = -\kappa \frac{\partial^2 T}{\partial x_i^2} - \frac{\partial T}{\partial x_i}\frac{\partial \kappa}{\partial x_i} \tag{3d-21a}$$

The last term, containing the gradient of κ, must be retained if implicit dependence of κ on the state variables is to be represented. On the other hand, if κ is assumed to be constant, (3c-21a) reduces to the more familiar form

$$\nabla \cdot \mathbf{q}_{\text{cond}} = -\kappa \nabla^2 T$$

The component of heat flux due to radiation can be approximated, for small temperature differences, by Newton's law of cooling,

$$\frac{\partial (q_i)_{\text{rad}}}{\partial x_i} = \rho C_v \mathfrak{q}(T - T_0) = \nabla \cdot \mathbf{q}_{\text{rad}} \tag{3c-21b}$$

where $(T - T_0)$ is the local temperature excess and $\mathfrak{q}$ is a radiation coefficient introduced by Stokes.[1] The foregoing thermal relations can be combined with the equations of continuity and momentum more readily if the term $T(Ds/Dt)$ appearing in (3c-19) is expressed in terms of the variables $\mathbf{u}$, v, and T. The defining equations (3c-14) establish that $P = P(v,s)$ and $T = T(v,s)$; from which it follows that one may also write $s = s(T,v)$ or $s = s(T,P)$. Using both of the latter leads, after some manipulation,[2] to the identity

$$\rho T \frac{Ds}{Dt} = \rho C_v \left[(\gamma - 1)\frac{\Delta}{\beta} + \frac{DT}{Dt} \right] \tag{3c-22}$$

in which β is the coefficient of thermal expansion, $\beta \equiv \rho(\partial v/\partial T)_P$. After (3c-22) and (3c-21) are combined with (3c-19), the energy equation can be written in the alternate forms

$$\frac{\rho C_v DT}{Dt} + \rho C_v \frac{\gamma - 1}{\beta}\frac{\partial u_i}{\partial x_i} + \frac{\partial q_i}{\partial x_i} - \phi_\eta = 0$$

$$\rho C_v \left(\frac{\partial T}{\partial t} + \mathbf{u}\cdot\nabla T\right) + \frac{\rho(C_p - C_v)}{\beta}\Delta - \nabla\cdot(\kappa \nabla T) + \rho C_v \mathfrak{q}(T - T_0) - \phi_\eta = 0 \tag{3c-23}$$

$$\frac{\partial T}{\partial t} + \mathbf{u}\cdot\nabla T + \frac{(\gamma - 1)}{\beta}\Delta - \frac{\kappa}{\rho C_v}\nabla^2 T - \frac{\nabla T \cdot \nabla \kappa}{\rho C_v} + \mathfrak{q}(T - T_0) - \frac{\phi_\eta}{\rho C_v} = 0$$

The viscous dissipation function ϕ_η can be evaluated, with the aid of (3c-8) and (3c-9) in the explicit form

$$\begin{aligned}\phi_\eta &= V_{ij}d_{ji} = \eta' d_{kk}d_{ii} + 2\eta d_{ij}d_{ji} \\ &= \eta_B \Delta^2 + \frac{4}{3}\eta \left[\left(\frac{\partial u_1}{\partial x_1}\right)^2 + \left(\frac{\partial u_2}{\partial x_2}\right)^2 + \left(\frac{\partial u_3}{\partial x_3}\right)^2 - \frac{\partial u_1}{\partial x_1}\frac{\partial u_2}{\partial x_2} - \frac{\partial u_2}{\partial x_2}\frac{\partial u_3}{\partial x_3} - \frac{\partial u_3}{\partial x_3}\frac{\partial u_1}{\partial x_1} \right] \\ &\quad + \eta \left[\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)^2 + \left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right)^2 + \left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\right)^2 \right]\end{aligned} \tag{3c-24a}$$

The thermal dissipation function ϕ_κ due to heat conduction can be evaluated, with the aid of (3c-16) and (3c-21a), in the form

$$\phi_\kappa = -\frac{q_i}{T}\frac{\partial T}{\partial x_i} = +\frac{\kappa}{T}\left(\frac{\partial T}{\partial x_i}\right)^2 = \frac{\kappa}{T}(\nabla T)^2 \tag{3c-24b}$$

It does not appear explicitly in (3c-23), but it is there implicitly as a consequence of the heat-transfer processes described by (3c-23).

[1] *Phil. Mag.* (4) **1**, 305–317 (1851).

[2] See, for example, Zemansky, "Heat and Thermodynamics," 3d ed., pp. 246–255, McGraw-Hill Book Company, Inc., New York, 1951.

Summary of Assumptions. The fluid considered is assumed to be continuous except at boundaries or interfaces, locally homogeneous and isotropic when at rest, viscous, thermally conducting, and chemically inert, and its local thermodynamic condition is assumed to be completely determined by specifying three "state" variables, any two of which determine the third uniquely through an equation of state. No structural or thermal "relaxation" mechanism has been presumed up to this point in the analysis, except to the extent that ordinary heat conduction and viscous losses may be described in such terms. Local thermodynamic reversibility has been assumed in using conventional thermodynamic identities based on the second law, but the irreversible production of entropy by dissipative processes has been accounted for explicitly. It is also assumed that the stress tensor is a linear function of the rate of deformation, and that the tractions due to viscosity can be represented by the linear terms of an expansion in powers of the viscosity coefficients. The viscosity and heat-exchange parameters of the fluid η, η', κ, and $\mathfrak{q}$, may depend in any continuous way on the state variables and hence may be implicit functions of time and the spatial coordinates. Within the scope thus defined the equations given are exact.

The functional dependence on time and the spatial coordinates of the condition and motion variables P, T, ρ, and $\mathbf{u}$ can be evaluated, in a formal sense at least, by solving the set of four simultaneous equations connecting these variables [Eqs. (3c-5), (3c-11), (3c-23), and (3c-15) or one of its alternates]. No general solution of these complete equations has been given, however, and one or another of the least important terms are usually omitted in order to render the equations tractable for dealing with specific problems.

3c-3. The Small-signal Acoustic Equations. The physical theory of sound waves deals with systematic motions of a material medium relative to an equilibrium state and thus comprises the variational aspects of elasticity and fluid dynamics. Such perturbations of state can be described by incremental, or acoustic, variables and approximate equations governing them can be obtained by arbitrarily "linearizing" the general equations of motion. These results, as well as higher-order approximations, can be derived in an orderly way by invoking a modified perturbation analysis.[1] This consists of replacing the dependent variables appearing in (3c-5), (3c-11), and (3c-23) by the sum of their equilibrium or zero-order values and their first- and second-order variational components, and then forming the separate equations that must be satisfied by the variables of each order. Two of the composite state variables, for example ρ and T, can be defined arbitrarily, whereupon the third, P, is determined by the functional equation of state. These definitions, some self-evident manipulations, and the subscript notation identifying the orders can be exhibited as follows:

$$\rho \equiv \rho_0 + \rho_1 + \rho_2 \qquad T \equiv T_0 + \theta_1 + \theta_2$$
$$\nabla\rho = \nabla\rho_1 + \nabla\rho_2 \qquad \nabla T = \nabla\theta_1 + \nabla\theta_2$$
$$P(\rho,T) \equiv P_0(\rho_0,T_0) + p_1 + p_2$$
$$p_1 + p_2 = \left[\left(\frac{\partial P}{\partial \rho}\right)_T\right]_0 (\rho - \rho_0) + \left[\left(\frac{\partial P}{\partial T}\right)_\rho\right]_0 (T - T_0) + \cdots \tag{3c-25}$$
$$K = K_T \equiv \rho\left(\frac{\partial P}{\partial \rho}\right)_T \qquad \beta \equiv -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_P \qquad c_0^2 \equiv \left[\left(\frac{\partial P}{\partial \rho}\right)_s\right]_0 \equiv \frac{(K_s)_0}{\rho_0}$$
$$\gamma = \frac{K_s}{K_T} = \frac{C_p}{C_v}$$
$$p_1 = \frac{c_0^2}{\gamma}(\rho_1 + \beta_0\rho_0\theta_1) \qquad p_2 = \frac{c_0^2}{\gamma}(\rho_2 + \beta_0\rho_0\theta_2)$$
$$\mathbf{u} \equiv 0 + \mathbf{u}_1 + \mathbf{u}_2 \qquad \nabla\cdot\mathbf{u} \equiv \Delta \equiv \Delta_1 + \Delta_2 = \nabla\cdot\mathbf{u}_1 + \nabla\cdot\mathbf{u}_2$$
$$\rho\mathbf{u} = [\rho_0\mathbf{u}_1]_1 + [\rho_1\mathbf{u}_1 + \rho_0\mathbf{u}_2]_2 + \cdots$$
$$\nabla\cdot(\rho\mathbf{u}) = [\rho_0\nabla\cdot\mathbf{u}_1]_1 + [\rho_1\nabla\cdot\mathbf{u}_1 + \mathbf{u}_1\cdot\nabla\rho_1 + \rho_0\nabla\cdot\mathbf{u}_2]_2 + \cdots$$

[1] Eckart, *Phys. Rev.* **73**, 68–76 (1948).

Terms containing $\nabla\rho_0$ have been omitted in writing out $\nabla \cdot (\rho\mathbf{u})$, on the assumption that ρ_0, T_0, and P_0 are constant and $\mathbf{u}_0 = 0$. The reference state need not be so restricted to one of static equilibrium provided its time and space rates of change are presumed small in comparison with the corresponding change rates of the acoustic variables. The extraneous body force $\mathbf{F}$ will also be omitted hereafter; it would become important in cases involving electromagnetic interaction, but it usually derives from a gravitation potential and affects primarily the equilibrium configuration.[1] Little generality is sacrificed by omitting $\mathbf{F}$ and assuming a static reference, moreover, since the basic equations characterize directly the equilibrium condition and since the "cross-modulation" effects brought in by nonlinearity are dealt with adequately through second- or higher-order approximations.

Notice that the foregoing represents a mathematical-approximation procedure that is concerned only with the *precision* achieved in interpreting the content of the basic equations. The *accuracy* with which the basic equations themselves delineate the behavior of a real fluid is an entirely different question that must be considered independently on its own merits. It follows that, while good judgment may restrain the effort, there is no impropriety involved in pursuing higher-order solutions of the acoustic equations, even though the equations themselves may embody first-order approximations to reality such as that represented by assuming linear dependence on the viscosity coefficients and the deformation rate.

When the appropriate relations from (3c-25) are substituted in (3c-5), (3c-11), and (3c-23), the *first-order acoustic equations* can be separated out in the form

$$\frac{\partial \rho_1}{\partial t} + \rho_0(\nabla \cdot \mathbf{u}_1) = 0 \tag{3c-26a}$$

$$\rho_0 \frac{\partial \mathbf{u}_1}{\partial t} + \frac{c_0^2}{\gamma}\left(1 + \beta_0\rho_0 \frac{\nabla\theta_1}{\nabla\rho_1}\right)\nabla\rho_1 - (\eta_0\mho)\nabla(\nabla \cdot \mathbf{u}_1) + \eta_0\nabla \times (\nabla \times \mathbf{u}_1) = 0 \tag{3c-26b}$$

$$\rho_0 C_v \frac{\partial \theta_1}{\partial t} + \frac{\rho_0 C_v(\gamma - 1)}{\beta_0}(\nabla \cdot \mathbf{u}_1) - \kappa_0\nabla^2\theta_1 + \rho_0 C_v \mathfrak{q}\theta_1 = 0 \tag{3c-26c}$$

Inasmuch as the first-order effects of both shear and dilatational viscosity and of heat conduction and radiation have been included, these equations comprehend a *visco-thermal theory* of small-signal sound waves. The sound absorption and velocity dispersion predicted by this theory are discussed below. Note especially that taking heat exchange into account explicitly by including (3c-26*c*) has precluded the conventional adiabatic assumption and denied the simplifying assumption that $P = P(\rho)$.

Adiabatic behavior would be assured, on the other hand, if it were assumed at the outset that $\kappa = \mathfrak{q} = 0$, but the behavior would *not* at the same time be strictly isentropic so long as irreversible viscous losses are still present and accounted for. The difference between adiabatic and isentropic behavior in this case is of second order, however, as indicated by the fact that the second-order dissipation functions ϕ do not appear in the first-order energy equation (3c-26*c*), which is thereby reduced to yielding just the isentropic relation between dilatation and excess temperature. It is allowable, therefore, in this first-order approximation, to replace the quotient $(\nabla\theta_1/\nabla\rho_1)$ appearing in (3c-26*b*) with the isentropic derivative $(\partial T/\partial\rho)_s = (\gamma - 1)/\rho\beta$, whereupon the first-order equation of motion for an *adiabatic viscous* fluid can be written as

$$\rho_0 \frac{\partial \mathbf{u}_1}{\partial t} + c_0^2\nabla\rho_1 - \eta_0\mho\nabla(\nabla \cdot \mathbf{u}_1) + 2\eta_0(\nabla \times \mathbf{R}_1) = 0 \tag{3c-27}$$

If the effects of viscosity, as well as of heat exchange, are to be neglected, the divergence of what is left of (3c-27) can be subtracted from the time derivative of (3c-26*a*)

[1] But, for a case in which $\mathbf{F}$ and $\nabla\rho_0$ cannot be neglected, see Haskell, *J. Appl. Phys.* **22**, 157–168 (February, 1951).

to yield the typical *small-signal scalar wave equation* of classical acoustics,

$$\frac{\partial^2 \rho_1}{\partial t^2} = \left(\frac{\partial P}{\partial \rho}\right)_s \nabla^2 \rho_1 \tag{3c-28a}$$

and, with the help of the first-order isentropic relation $p_1 = c_0^2(\rho_1)_s$, this wave equation becomes, in terms of the sound pressure,

$$\frac{\partial^2 p_1}{\partial t^2} = c_0^2 \nabla^2 p_1 \tag{3c-28b}$$

3c-4. The Second-order Acoustic Equations. The same substitution of composite variables that delivered (3c-26*a*), (3c-26*b*), and (3c-26*c*) will also yield directly the second-order equations of acoustics, which can now be marshaled as follows:

$$\frac{\partial \rho_2}{\partial t} + \rho_0(\nabla \cdot \mathbf{u}_2) + \nabla \cdot (\rho_1 \mathbf{u}_1) = 0 \tag{3c-29a}$$

$$\begin{aligned} \rho_0 \frac{\partial \mathbf{u}_2}{\partial t} + \frac{\partial(\rho_1 \mathbf{u}_1)}{\partial t} &+ \rho_0 \mathbf{u}_1(\nabla \cdot \mathbf{u}_1) + (\mathbf{u}_1 \cdot \nabla)\mathbf{u}_1 \\ &+ \frac{c_0^2}{\gamma}\left(1 + \beta_0 \rho_0 \frac{\nabla \theta_2}{\nabla \rho_2}\right) \nabla \rho_2 - \eta_0 \mho \nabla(\nabla \cdot \mathbf{u}_2) + 2\eta_0(\nabla \times \mathbf{R}_2) \\ &- (\nabla \eta_1')(\nabla \cdot \mathbf{u}_1) - 2(\nabla \eta_1 \cdot \nabla)\mathbf{u}_1 - 2(\nabla \eta_1) \times \mathbf{R}_1 = 0 \end{aligned} \tag{3c-29b}$$

$$\begin{aligned} \frac{\partial \theta_2}{\partial t} + \mathbf{u}_1 \cdot (\nabla \theta_1) &+ \frac{\gamma - 1}{\beta_0}(\nabla \cdot \mathbf{u}_2) - \frac{\kappa_0}{\rho_0 C_v} \nabla^2 \theta_2 \\ &+ \frac{\kappa_0}{\rho_0^2 C_v} \rho_1 \nabla^2 \theta_1 - \frac{\nabla \theta_1 \cdot \nabla \kappa_1}{\rho_0 C_v} + \mathfrak{q}\theta_2 - \frac{\phi_\eta}{\rho_0 C_v} = 0 \end{aligned} \tag{3c-29c}$$

The subscripts appended to κ and the η's imply that each may be expressed in the generic form

$$\eta(T, \rho, \cdot \cdot \cdot) = \eta_0(T_0, \rho_0, \cdot \cdot \cdot) + \eta_1 \qquad \eta_1 = \frac{\partial \eta}{\partial T}\theta_1 + \frac{\partial \eta}{\partial \rho}\rho_1 + \cdot \cdot \cdot \tag{3c-30}$$

No general solution of these complete second-order equations has been given, but they provide a useful point of departure for making approximations and for investigating some second-order phenomena that cannot be predicted by the first-order equations alone.

3c-5. Spatial and Material Coordinates. Equations (3c-26) and (3c-29) are couched in terms of the local values assumed by the dependent variables ρ, P, T, and $\mathbf{u}$ at *places* identified by their coordinates x_i in a fixed *spatial* reference frame, commonly called *Eulerian* coordinates (in spite of their first use by d'Alembert). As an alternate method of representation, the behavior of the medium can be described in terms of the sequence of values assumed by the dependent condition and state variables pertaining to identified *material* particles of the medium no matter how these particles may move with respect to the spatial coordinate system. The independent variables in this case are the identification coordinates a_i, rather than the position coordinates; the latter then become dependent variables that describe, as time progresses, the travel history of each particle of the medium. Such a representation in terms of *material* coordinates is commonly called *Lagrangian* (in spite of its first introduction and use by Euler).

The Wave Equation in Material Coordinates. The use of material coordinates can be demonstrated by deriving the exact equations governing one-dimensional (plane-wave) propagation in a *nonviscous adiabatic* fluid. Consider a cylindrical segment of the medium of unit cross section with its axis along $+x$, the direction of propagation, and let x and $x + \delta x$ define the boundaries of a thin laminar "particle" whose undisturbed equilibrium position is given by a and $a + \delta a$. The difference $x - a = \xi$ defines the displacement of the a particle from its equilibrium position and provides a convenient incremental, or acoustic, dependent variable in terms of which to describe the position, velocity, and acceleration of the particle; thus

$$x(a,t) = a + \xi(a,t) \qquad \frac{\partial x}{\partial t} = u^L(a,t) = \frac{\partial \xi}{\partial t} \qquad \frac{\partial u^L}{\partial t} = \frac{\partial^2 \xi}{\partial t^2} \tag{3c-31}$$

Continuity requires that the mass of the particle remain constant during any displacement, which means that

$$\rho_0 \delta a = \rho^L \delta x = \rho^L \left(\delta a + \frac{\partial \xi}{\partial a} \delta a \right) \qquad \frac{\rho_0}{\rho^L} = \frac{\partial x}{\partial a} = 1 + \frac{\partial \xi}{\partial a} \tag{3c-32a}$$

or, for three-dimensional disturbances and in general,

$$\frac{\rho_0}{\rho^L} = \frac{\partial(x_1,x_2,x_3)}{\partial(a_1,a_2,a_3)} \tag{3c-32b}$$

in which the symbolic derivative stands for the Jacobian functional determinant. The superscript L is used here and below as a reminder that the dependent variable so tagged adheres to, or "follows" in the Lagrangian sense, a specific particle, and that it is a function of the independent identification coordinates. When not so tagged, or with superscript E added for emphasis, the state variables ρ, P, T, and the condition variable u are each assumed to be functions of time and the spatial coordinate x.

The net force per unit mass acting on the particle at time t is $-(\rho^L)^{-1}\partial P^L/\partial x$, where ρ^L and P^L are the density and pressure at x, the "now" position of the moving particle. However, inasmuch as x is not an independent variable in this case, the pressure gradient must be rewritten as $(\partial P^L/\partial a)(\partial a/\partial x)$, from which the second factor can be eliminated by recourse to (3c-32a). The momentum equation then becomes just

$$\frac{\rho_0 \partial^2 \xi}{\partial t^2} = \frac{-\partial P^L}{\partial a} \tag{3c-33}$$

The adiabatic assumption makes available the simplified equation of state, $P = P(\rho)$, and this relation, in turn, allows the material gradient, $\partial P^L/\partial a$, to be written as

$$\frac{-\partial P^L}{\partial a} = -\left(\frac{\partial P^L}{\partial \rho^L}\right)_s \frac{\partial \rho^L}{\partial a} = -c^2 \frac{\partial \rho^L}{\partial a} \tag{3c-34}$$

from which the last factor can be eliminated by using (3c-32a) again. This leads at once to the exact wave equation[1]

$$\frac{\partial^2 \xi}{\partial t^2} = \left(\frac{c\rho^L}{\rho_0}\right)^2 \frac{\partial^2 \xi}{\partial a^2} = c^2 \left(1 + \frac{\partial \xi}{\partial a}\right)^{-2} \frac{\partial^2 \xi}{\partial a^2} \tag{3c-35}$$

The pressure-density relation for a perfect adiabatic gas is $P = P_0(\rho/\rho_0)^\gamma$, from which it can be deduced that

$$c^2 = \left(\frac{\partial P}{\partial \rho}\right)_s = \frac{\gamma P_0}{\rho_0}\left(\frac{\rho}{\rho_0}\right)^{\gamma-1} = c_0^2 \left(\frac{\rho}{\rho_0}\right)^{\gamma-1} \tag{3c-36}$$

No generalization of comparable simplicity is available for liquids.[2] When (3c-36) is introduced in (3c-35), the exact "Lagrangian" wave equation for an adiabatic perfect gas becomes

$$\frac{\partial^2 \xi}{\partial t^2} = c_0^2 \left(\frac{\rho^L}{\rho_0}\right)^{\gamma+1} \frac{\partial^2 \xi}{\partial a^2} = c_0^2 \left(1 + \frac{\partial \xi}{\partial a}\right)^{-(\gamma+1)} \frac{\partial^2 \xi}{\partial a^2} \tag{3c-37}$$

In the Lagrangian formulation illustrated above, the choice of a, the initial-position coordinate, as the independent variable is useful but any other coordinate that

[1] Rayleigh, "Theory of Sound" vol. II, §249; Lamb, "Hydrodynamics" §§13–15, 279–284.

[2] But see Courant and Friedrichs, "Supersonic Flow and Shock Waves," p. 8, Interscience Publishers, Inc., New York, 1948.

identifies the particles would serve the same purpose. For example, the particle located momentarily at x can be uniquely identified by the material coordinate $h \equiv \int_0^x \rho\, dx$, where h represents the mass of fluid contained between the origin and the particle. Inasmuch as this included mass will not change as the particle moves, the use of h as an independent "mass" variable automatically satisfies the requirements of continuity, with some attendant simplification in the analysis of transient disturbances. In the undisturbed condition, $\rho = \rho_0$ and $x = a$, whence the relation $a = h/\rho_0$ allows the independent variables to be interchanged by direct substitution in (3c-37).

Material and Spatial Coordinate Transforms. It is useful to have available a systematic procedure for converting a functional expression for one of the state variables from the form involving material coordinates to the corresponding form in spatial coordinates, or the inverse. One should avoid, however, the trap of referring to the state variables themselves as Lagrangian or Eulerian *quantities;* density and pressure, for example, are scalar point functions that can have only one value at a given place and time. On the other hand, it is of prime importance to distinguish carefully (and to specify!) the independent variables when computing the derivatives of these quantities.

The E and L functions are tied together by the displacement variable ξ, which provides a single-valued connection between the a particle and its instantaneous position coordinate x and which may therefore be regarded as a function of either of its terminal coordinates a or x. This can be indicated [cf. (3c-31)] by writing $x(a,t) = a + \xi(a,t)$, or the inverse relation $a(x,t) = x - \xi(x,t)$; from which follow the alternate expressions

$$a = x - \xi(a,t) \qquad x = a + \xi(x,t) \tag{3c-38}$$

The desired coordinate transforms can then be established by means of Taylor series expansions, the two forms following according to whether the expansion is centered on the instantaneous particle position or spatial coordinate x, or on the particle's equilibrium position or material coordinate a. Thus, if q is used to represent any one of the variables ρ, P, T, or u, one of the expansions can be based on the obvious identity

$$\begin{aligned} q^L(a,t) &= q^E(x,t)_{x=a+\xi(x,t)} \\ &= q^E(x,t)_{x=a} + \left[\xi(x,t)\,\frac{\partial q^E(x,t)}{\partial x}\right]_{x=a} + \frac{1}{2}\left[\xi^2(x,t)\,\frac{\partial^2 q^E(x,t)}{\partial x^2}\right]_{x=a} + \cdots \end{aligned} \tag{3c-39}$$

Note that all terms on the right of (3c-39) are functions of the spatial coordinates and that each is to be evaluated at the equilibrium position coordinate a. This transform yields, therefore, the instantaneous value in material coordinates of the variable represented by q, in terms of the local value of q modified by correction terms (comprising the succeeding terms of the series) based on the spatial rate of change of q and the instantaneous displacement.

The inverse transform is derived in a similar way from the identity

$$\begin{aligned} q^E(x,t) &= [q^L(a,t)]_{a=x-\xi(a,t)} \\ q^E(x,t) &= [q^L(a,t)]_{a=x} - \left[\xi(a,t)\,\frac{\partial q^L(a,t)}{\partial a}\right]_{a=x} + \frac{1}{2}\left[\xi^2(a,t)\,\frac{\partial^2 q^L(a,t)}{\partial a^2}\right]_{a=x} - \cdots \end{aligned} \tag{3c-40}$$

In symmetrical contrast with (3c-39), all terms on the right in (3c-40) are functions of the material coordinates and are to be evaluated for $a = x$. This transform, therefore, yields the instantaneous local value of the variable q at the place x, in terms of the instantaneous value of q for the now-displaced particle whose equilibrium position or material coordinate is $a = x$, modified by the succeeding terms of the series in accordance with the material-coordinate rate of change of q and the instantaneous displacement.

The transforms (3c-39) and (3c-40) indicate that the differences between q^L and q^E are of second order, which explains why the troublesome distinction between spatial and material coordinates does not intrude when only first-order effects are being considered. It also follows that the first two terms of these transforms are sufficient to deliver all terms of q^L or q^E through the second order. The use of these transforms can be illustrated by writing them out explicitly for u and ρ, including all second-order terms;

$$u^L \equiv \xi_t \qquad u^E = u^L - \xi u_a{}^L = \xi_t - \xi\xi_{ta} \tag{3c-41a}$$

$$\rho^L = \rho_0(1 + \xi_a)^{-1} = \rho_0(1 - \xi_a + \xi_a{}^2 - \cdots)$$

$$\rho^E = \rho_0(1 - \xi_a + \xi_a{}^2 + \xi\xi_{aa}) = \rho_0[1 - \xi_a + (\xi\xi_a)_a] \tag{3c-41b}$$

in which the subscripts indicate partial differentiation with respect to a or t. The product of (3c-41a) and (3c-41b) gives at once the relation between the material and spatial coordinate expressions for the mass transport ρu; thus, through second order,

$$\rho^E u^E = \rho^L u^L - \xi(\rho^L u^L)_a + \xi^2(\rho_a{}^L u_a{}^L) = \rho_0[\xi_t - (\xi\xi_t)_a] = \rho_0[\xi - \xi\xi_a]_t \tag{3c-42}$$

It is then straightforward to show that, if the particle velocity ξ_t is simple harmonic, the time average of the local mass transport $\rho^E u^E$ will vanish through the second order, even though the average value of u^E is not zero. Note, however, that the displacement velocity ξ_t is measured from an equilibrium position that is here assumed to be static; the average mass transport may indeed take on nonvanishing values if the wave motion as a whole leads to gross streaming (see Sec. 3c-7).

3c-6. Waves of Finite Amplitude. A distinguished tradition adheres to the study of the propagation of unrestricted compressional waves. That the particle velocity is forwarded more rapidly in the condensed portion of the wave was known early (Poisson, 1808; Earnshaw, 1858; Riemann, 1859); and that this should lead eventually to the formation of a discontinuity or shock wave was recognized by Stokes (1848), interpreted by Rayleigh,[1] discussed more recently by Fubini,[2] and has been reviewed still more recently with heightened interest by modern students of blast-wave transmission.[3]

By virtue of the adiabatic assumption underlying $P = P(\rho)$, the speed of sound is also a function of density alone and may be approximated by the leading terms of its expansion about the equilibrium density:

$$c^2 \doteq c_0{}^2\left[1 - 2\xi_a \frac{\rho_0}{c_0}\left(\frac{Dc}{D\rho}\right)_0 + \cdots\right] \tag{3c-43}$$

When (3c-43) is introduced in the exact wave equation in material coordinates, (3c-35), the latter can be recast in the following form, using the subscript convention for partial differentiation and retaining only, but all, terms through second order:

$$\xi_{tt} - c_0{}^2\xi_{aa} = -c_0{}^2\left[1 + \frac{\rho_0}{c_0}\left(\frac{Dc}{D\rho}\right)_0\right](\xi_a{}^2)_a \tag{3c-44}$$

If it is then assumed that an arbitrary plane displacement $\xi(0,t) = f(t)$ is impressed at the origin, it can be verified by direct substitution that a solution of (3c-44) is

$$\xi(a,t) = f\left(t - \frac{a}{c_0}\right) + \frac{a}{2c_0{}^2}\left[1 + \frac{\rho_0}{c_0}\left(\frac{Dc}{D\rho}\right)_0\right]\left[f'\left(t - \frac{a}{c_0}\right)\right]^2 \tag{3c-45}$$

The density variations associated with these displacements are to be found by entering (3c-45) in (3c-32), and the variational pressure can then be evaluated in terms of the adiabatic compressibility of the medium.

Relatively more attention has been devoted to the analysis of solutions of (3c-37) for the case of an adiabatic perfect gas. For an arbitrary initial displacement, as

[1] "Theory of Sound," vol. II, §§249–253.

[2] *Alta Frequenza* **4**, 530–581 (1935).

[3] See also Sec. 2z of this book, "Shock Waves," pp. 2-231 to 2-236.

above, the solution of the corresponding wave equation (3c-37), again including all terms through second order, is

$$\xi(a,t) = f\left(t - \frac{a}{c_0}\right) + \frac{a}{2c_0^2}\frac{\gamma + 1}{2}\left[f'\left(t - \frac{a}{c_0}\right)\right]^2 \tag{3c-46}$$

Technological interest in this problem centers on the generation of spurious harmonics, which can be studied by assuming the initial displacement to be simple harmonic, viz., $f(t) = \xi_0 (1 - \cos \omega t)$ at the origin. The solution then takes the explicit form

$$\xi(a,t) = \xi_0 [1 - \cos (\omega t - k_0 a)] + \frac{\gamma + 1}{8} k_0^2 \xi_0^2 a [1 - \cos 2(\omega t - k_0 a)] \tag{3c-47}$$

in which k_0 is written for the phase constant, $k_0 = \omega/c_0 = 2\pi/\lambda_0$.

The most striking feature of the solutions (3c-45) and (3c-47) is the appearance of the material coordinate a in the coefficient of the second-harmonic term. As a consequence, the condensation wave front becomes progressively steeper as the wave propagates, the energy supplied at fundamental frequency being gradually diverted toward the higher harmonic components. The compensating diminution of the fundamental-frequency component would be exhibited explicitly if third-order terms had been retained in (3c-46) and (3c-47) inasmuch as all odd-order terms include a "contribution" to the fundamental. When such higher terms are retained it is predicted that propagation will always culminate in the formation of a shock wave at a distance from the source given approximately by $a \doteq 2\xi_0/(\gamma + 1)M^2$, where M is the peak value of the particle-velocity Mach number.[1] On the other hand, when dissipative mechanisms are taken into account, the fact that attenuation increases with frequency for either liquids or gases leads to the result that, except for very large initial disturbances, a stable value of wave-front steepness will be reached at which the rate of energy conversion to higher frequencies by nonlinearity is just compensated by the increase of absorption at higher frequencies. If attention is centered on the fundamental component, however, such diversion of energy to higher frequencies appears as an attenuation and accounts for the relatively more rapid absorption sometimes observed near a sound source.[2]

The variational or acoustic pressure, in material coordinates, can be expressed generally as a function of the displacement gradients by using the adiabatic pressure-density relation $P^L = P_0(\rho^L/\rho_0)^\gamma$ in conjunction with the continuity relation (3c-32); thus,

$$P^L - P_0 = p^L = \gamma P_0[-\xi_a + \tfrac{1}{2}(\gamma + 1)\xi_a^2] = \langle p^L \rangle + p_1^L + p_2^L \tag{3c-48}$$

in which the last member identifies the steady-state alteration of the average pressure and the fundamental and second-harmonic components of sound pressure. When the harmonic solution (3c-47) is introduced in (3c-48), the two alternating components of pressure for $a^2 \gg (\lambda/4\pi)^2$ can be shown, after some algebraic manipulation, to be

$$p_1^L = +\gamma P_0 M \sin (\omega t - k_0 a) = +\sqrt{2}\, P_1 \sin (\omega t - k_0 a) \tag{3c-49a}$$

$$p_2^L = \gamma P_0 M^2 k_0 a \tfrac{1}{4}(\gamma + 1) \sin 2(\omega t - k_0 a) = \sqrt{2}\, P_2 \sin 2(\omega t - k_0 a) \tag{3c-49b}$$

in which P_1 and P_2 are the rms values of the fundamental and second-harmonic sound pressures, and $M = k_0\xi_0 = \omega\xi_0/c_0$ is again the peak value of the particle-velocity Mach number at the origin. The relative magnitude of P_2 increases linearly with distance from the origin and is directly proportional to the peak Mach number, as may be deduced from (3c-49a) and (3c-49b); thus

$$\frac{P_2}{P_1} = \frac{1}{4}(\gamma + 1)Mk_0 a \qquad P_2 = \frac{P_1^2 k_0 a(\gamma + 1)}{2\sqrt{2}\,\gamma P_0} \tag{3c-50}$$

[1] Fubini, *Alta Frequenza* **4**, 530–581 (1935).
[2] Fox and Wallace, *J. Acoust. Soc. Am.* **26**, 994–1006 (1954).

Various experimental studies of second-harmonic generation have given results in reasonably good agreement with the predictions of (3c-50).[1]

The sound-induced alteration of mean total pressure, or "average" acoustic pressure, is given by the time-independent terms yielded by the substitution of (3c-47) in (3c-48), viz.,

$$\langle p^L \rangle = + \frac{\gamma P_0 M^2(\gamma + 1)}{8} \tag{3c-51}$$

Note that this pressure increment is given as a function of the material coordinates, which means that it pertains to a *moving* element of the fluid. The *local* value of the pressure change can be found by means of the transform (3c-40), which gives, through second-order terms, the following replacement for (3c-48);

$$p^E = p^L - \xi \frac{\partial p^L}{\partial a} = \gamma P_0 \left[-\xi_a + \frac{1}{2}(\gamma + 1)\xi_a{}^2 + \xi\xi_{aa} \right] \tag{3c-52}$$

When (3c-47) is introduced in (3c-52), the time-independent terms give the local change in mean pressure as

$$\langle p^E \rangle = + \frac{\gamma P_0 M^2(\gamma - 3)}{8} \tag{3c-53}$$

and since γ is usually less than 2, it follows that the local value of mean pressure will be *reduced* by the presence of the sound wave, in striking contrast to the *increase* of mean pressure that would be observed when following the motion of a particle of the medium. Negative pressure increments as large as 10 newtons m^{-2} (100 dynes cm^{-2}) have been reported experimentally, in reasonably good agreement with (3c-53).

The mean value of the material particle velocity, $u^L \equiv \xi_t$, vanishes, as may be seen by differentiating (3c-47). The local particle velocity that would be observed at a fixed spatial position does not similarly vanish, however, and may be shown, by using the transform (3c-40) again, to be

$$u^E = \xi_t - \xi\xi_{ta} \qquad \langle u^E \rangle = -\frac{1}{2} c_0 M^2 = -\frac{\rho_0 c_0 \omega^2 \xi_0{}^2}{2\rho_0 c_0{}^2} = -(\rho_0 c_0{}^2)^{-1}\langle J \rangle \tag{3c-54}$$

where $\langle J \rangle$ is the average sound energy flux, or sound intensity.[2]

3c-7. Vorticity and Streaming. As suggested above, and with scant respect for the traditional symmetry of simple-harmonic motion, sound waves are found experimentally to exert net time-independent forces on the surfaces on which they impinge, and there is often aroused in the medium a pattern of steady-state flow that includes the formation of streams and eddies. The exact wave equation considered in the preceding section has been solved only for one-parameter waves (i.e., plane or spherical), and these solutions do not embrace some of the gross rotational flow patterns that are observed to occur. It is necessary, therefore, to revert for the study of these phenomena to the perturbation procedures introduced by the first- and second-order equations (3c-26) and (3c-29).

It is plausible that vortices and eddies should arise, if there is any net transport at all, inasmuch as material continuity would require that any net flow in the direction of sound propagation must be made good in the steady state by recirculation toward the source. Streaming effects can be studied most usefully, therefore, in terms of the generation and diffusion of circulation, or vorticity. More specifically, the time average of the second-order velocity $\mathbf{u}_2$ will be a first-order measure of the streaming

[1] Thuras, Jenkins, and O'Neil, *J. Acoust. Soc. Am.* **6**, 173–180 (1935); Fay, *J. Acoust. Soc. Am.* **3**, 222–241 (October, 1931); O. N. Geertsen, unpublished (ONR) Tech. Report no. III, May, 1951, U.C.L.A.

[2] Westervelt, *J. Acoust. Soc. Am.* **22**, 319–327 (1950).

velocity. The vector function describing $\mathbf{u}_2$ can always be resolved into solenoidal and lamellar components defined by

$$\mathbf{u}_2 \equiv -\nabla\varphi_2 + \nabla \times \mathbf{A}_2 \qquad \nabla^2\varphi_2 \equiv -\nabla \cdot \mathbf{u}_2 \qquad \nabla^2\mathbf{A}_2 = -(\nabla \times \mathbf{u}_2) \quad \text{(3c-55)}$$

The irrotational component that represents the compressible, or acoustic, part of the fluid motion is derived from the scalar potential φ_2. The vector potential $\mathbf{A}_2$ is associated with the rotational component comprising the incompressible circulatory flow that is of primary interest in streaming phenomena.

The failure of the first-order equations to predict streaming can be demonstrated by writing directly the curl of the first-order force equation (3c-26*b*). The gradient terms are eliminated by this operation, since $\nabla \times \nabla(\ ) \equiv 0$, leaving just

$$\frac{\partial \mathbf{R}_1}{\partial t} - \nu_0\nabla^2\mathbf{R}_1 = 0 \quad \text{(3c-56)}$$

Thus the first-order vorticity, $\mathbf{R}_1 \equiv \frac{1}{2}(\nabla \times \mathbf{u}_1)$, if it has any value other than zero, obeys a typical homogeneous diffusion equation. On the other hand, it would appear to follow that, if $\mathbf{R}_1$ were ever zero everywhere, its time derivative would also vanish everywhere and $\mathbf{R}_1$ would be constrained always thereafter to remain zero. This is *not* a valid proof of the famous Lagrange-Cauchy proposition on the permanence of the irrotational state, but the absence of any source terms on the right-hand side of (3c-56) does indicate correctly[1] that first-order vorticity cannot be generated in the interior of a fluid even when viscosity and heat conduction are taken into account. Instead, first-order vorticity, if it exists at all, must diffuse inward from the boundaries under control of (3c-56).

A notably different result is obtained when the second-order equations are dealt with in the same way. It is useful, before taking the curl of (3c-29*b*), to eliminate the second and third terms of this equation by subtracting from it the product of (ρ_1/ρ_0) and (3c-26*b*), and the product of $\mathbf{u}_1$ and (3c-26*a*). In effect this raises the first-order equations to second order and then combines the information in both sets. The augmented second-order force equation can then be arranged in the form

$$\rho_0\frac{\partial \mathbf{u}_2}{\partial t} + 2\eta_0(\nabla \times \mathbf{R}_2) + \nu_0\mho\rho_1\nabla(\nabla \cdot \mathbf{u}_1) - 2\nu_0\rho_1(\nabla \times \mathbf{R}_1) - 2\rho_0(\mathbf{u}_1 \times \mathbf{R}_1)$$
$$-2[(\nabla\eta_1 \cdot \nabla)\mathbf{u}_1 + \nabla\eta_1 \times (\nabla \times \mathbf{u}_1)] + 2(\nabla\eta_1 \times \mathbf{R}_1) + \rho_0\nabla\left(\frac{1}{2}\mathbf{u}_1 \cdot \mathbf{u}_1\right) + B_2\nabla\rho_2$$
$$- B_1\nabla\left(\frac{1}{2}\rho_1^2\right) - \eta_0\mho\nabla(\nabla \cdot \mathbf{u}_2) - \nabla\eta_1'(\nabla \cdot \mathbf{u}_1) = 0 \quad \text{(3c-57)}$$

The following abbreviations have been used for the coefficients of $\nabla\rho_1$ in (3c-26*b*) and of $\nabla\rho_2$ in (3c-29*b*):

$$B_1 \equiv \frac{c_0^2}{\gamma}\left[1 + \beta_0\rho_0\left(\frac{D\theta_1}{D\rho_1}\right)_0\right] \qquad B_2 \equiv \frac{c_0^2}{\gamma}\left[1 + \beta_0\rho_0\left(\frac{D\theta_2}{D\rho_2}\right)_0\right] \quad \text{(3c-58)}$$

in which the quotients $(\nabla\theta_1/\nabla\rho_1)$ and $(\nabla\theta_2/\nabla\rho_2)$ have been replaced by the corresponding material derivatives $D\theta/D\rho$, which must be evaluated, of course, for the particular conditions of heat exchange satisfying the energy equations (3c-26*c*) and (3c-29*c*). This evaluation can be evaded temporarily (at the cost of neglecting ∇B_1 and ∇B_2) by observing that each of the last five terms of (3c-57) contains a gradient. These disappear on taking the curl of (3c-57), whereupon the vorticity equation emerges as

$$\frac{\partial \mathbf{R}_2}{\partial t} - \nu_0\nabla^2\mathbf{R}_2 = \frac{1}{2}\nu_0\mho\left(\nabla s_1 \times \nabla\frac{\partial s_1}{\partial t}\right) + \rho_0^{-1}\nabla \times (\mathbf{u}_1 \cdot \nabla)\nabla\eta_1 + \nu_0 s_1\nabla^2\mathbf{R}_1$$
$$- \nu_0\nabla s_1 \times (\nabla \times \mathbf{R}_1) - \nabla \times (\mathbf{u}_1 \times \mathbf{R}_1) + \rho_0^{-1}\nabla \times (\nabla\eta_1 \times \mathbf{R}_1) \quad \text{(3c-59)}$$

[1] St. Venant, *Compt. rend.* **68**, 221–237 (1869).

in which s_1 has been introduced as an abbreviation for the first-order condensation, $s_1 = \rho_1/\rho_0$. This inhomogeneous diffusion equation puts in evidence various second-order sources of vorticity: four vanish if the first-order motion is irrotational ($\mathbf{R}_1 = 0$), and two drop out when the shear viscosity is constant ($\nabla\eta_1 = 0$). It is notable that the dilatational viscosity η' does not appear in any of these source terms except through the ratio η'/η that forms part of the dimensionless viscosity number $\mathcal{V} \equiv 2 + (\eta'/\eta)$.

Except for the third source term, which (3c-56) shows to be one order smaller than the change rate of $\mathbf{R}_1$, *all* the vorticity sources would vanish—and the streaming would "stall"—if the wave front were strictly plane with $\mathbf{u}_1$, s_1, and η functions of only one space coordinate. Wave fronts cannot remain strictly plane at grazing incidence, however,[1] and rapid changes in the direction and magnitude of $\mathbf{u}_1$ will occur near reflecting surfaces, in the neighborhood of sound-scattering obstacles, and in thin viscous boundary layers. As a consequence, the "surface" source terms containing $\mathbf{R}_1$ become relatively more important in these cases.[2] In other circumstances, when the sound field is spatially restricted by source directionality, the first source term in (3c-59) dominates and leads to a steady-state streaming velocity proportional to the *ratio* of the dilatational and shear viscosity coefficients—and hence to a unique independent method of measuring this moot ratio.[3] Both the force that drives the fluid circulation and the viscous drag that opposes it are proportional to the kinematic viscosity, which does not therefore control the final value of streaming velocity but only the time constant of the motion, i.e., the time required to establish the steady state.[4]

Evaluating the second-order vorticity source terms in any specific case requires that the first-order velocity field be known, and this calls in the usual way for solutions that satisfy the experimental boundary conditions and the wave equation. Unusual requirements of exactness are imposed on such solutions, moreover, by the fact that even the second-order acoustic equations yield only a first approximation to the mean particle velocity.

The analysis of vorticity can be recast, by skillful abbreviation and judicious regrouping of the elements of (3c-57), in such a way as to yield a general law of rotational motion, according to which the average rate of increase of the moment of momentum of a fluid element responds to the difference between the sound-induced torque and a viscous torque arising from the induced flow.[5] A close relation has also been shown to exist in some cases between the streaming potential and the attenuation of sound by the medium without regard for whether the attenuation is caused by viscosity, heat conduction, or by some relaxation process; in effect the average momentum of the stream "conserves" the momentum diverted from the sound wave by absorption.[6] This principle has so far been established rigorously only for the adiabatic assumption under which $P = P(\rho)$, and under restrictive assumptions on the variability of η and $\mathcal{V}$, but its prospective importance would appear to justify efforts to extend the generalization.

3c-8. Acoustical Energetics and Radiation Pressure. If the kinetic energy density that appeared briefly in (3c-12) is restored to (3c-18), the change rate of the specific

[1] Morse, "Vibration and Sound," 2d ed., pp. 368–371, McGraw-Hill Book Company, Inc., New York, 1948.

[2] Medwin and Rudnick, *J. Acoust. Soc. Am.* **25,** 538–540 (1953).

[3] Liebermann, *Phys. Rev.* **75,** 1415–1422 (1949); Medwin, *J. Acoust. Soc. Am.* **26,** 332–341 (1954).

[4] Eckart, *Phys. Rev.* **73,** 68–76 (1948).

[5] Nyborg, *J. Acoust. Soc. Am.* **25,** 938–944 (1953); Westervelt, *J. Acoust. Soc. Am.* **25,** 60–67 and *errata,* 799 (1953).

[6] Nyborg, *J. Acoust. Soc. Am.* **25,** 68–75 (1953); Doak, *Proc. Roy. Soc. (London),* ser. A, **226,** 7–16 (1954); Piercy and Lamb, *Proc. Roy. Soc. (London),* ser. A, **226,** 43–50 (1954).

total energy density (per unit mass), E/ρ, can be formulated in terms of

$$\rho \frac{D(E/\rho)}{Dt} = \rho \frac{D(\frac{1}{2}\mathbf{u} \cdot \mathbf{u})}{Dt} + \rho \frac{D\epsilon}{Dt}$$

$$= \rho \frac{D(\frac{1}{2}\mathbf{u} \cdot \mathbf{u})}{Dt} - \rho P \frac{Dv}{Dt} - \nabla \cdot \mathbf{q} + \phi_\eta \qquad (3c\text{-}60)$$

Material derivatives are used here so that the energy balance reckoned for a particular volume element will continue to hold as the derivatives "follow" the motion of the material particles. The mechanical work term on the right in (3c-60) can be resolved into two components by writing $P = P_0 + p$, where the excess, or sound, pressure p now represents the sum of the variational components of all orders

$$(p = p_1 + p_2 + \cdots)$$

Thus

$$\rho \frac{D(E/\rho)}{Dt} = \rho \frac{D(\frac{1}{2}\mathbf{u} \cdot \mathbf{u})}{Dt} - \rho p \frac{Dv}{Dt} + \rho P_0 \frac{Dv}{Dt} - \nabla \cdot \mathbf{q} + \phi_\eta \qquad (3c\text{-}61)$$

A second equation involving the first two terms on the right of (3c-61) can be formed by multiplying the continuity equation (3c-5) by p and adding it to the scalar product of the vector $\mathbf{u}$ and the vector force equation (3c-11b); thus

$$\rho \mathbf{u} \cdot \frac{D\mathbf{u}}{Dt} + \mathbf{u} \cdot \nabla p + p\left(\frac{1}{\rho}\frac{D\rho}{Dt} + \nabla \cdot \mathbf{u}\right) = \mathbf{u} \cdot \mathbf{f}_v(\eta, \eta', \mathbf{u})$$

$$= \rho \mathbf{u} \cdot \frac{D\mathbf{u}}{Dt} - p\rho \frac{Dv}{Dt} + \mathbf{u} \cdot \nabla p + p \nabla \cdot \mathbf{u} \qquad (3c\text{-}62)$$

where $\mathbf{f}_v$ stands for the sum of the five viscosity terms that appear on the right-hand side of (3c-11b). Combining this result with (3c-61) gives

$$\rho \frac{D(\frac{1}{2}\mathbf{u} \cdot \mathbf{u})}{Dt} - \rho p \frac{Dv}{Dt} + \nabla \cdot (p\mathbf{u}) = +\mathbf{u} \cdot \mathbf{f}_v$$

$$\rho \frac{D(E/\rho)}{Dt} + \nabla \cdot (p\mathbf{u}) = -\rho P_0 \frac{Dv}{Dt} - \nabla \cdot \mathbf{q} + \phi_\eta + \mathbf{u} \cdot \mathbf{f}_v \qquad (3c\text{-}63)$$

The significance of this result can be made more apparent by using the continuity equation again, this time in the form $(E/\rho)[\partial\rho/\partial t + \nabla \cdot (\rho\mathbf{u})] = 0$. Adding this "zero" to the left-hand side of (3c-63), after first using (3c-3) to express the material derivative in terms of fixed spatial coordinates, allows the continuity of acoustic energy to be expressed by

$$\rho \frac{D(E/\rho)}{Dt} + \nabla \cdot (\rho\mathbf{u}) = \rho \frac{\partial(E/\rho)}{\partial t} + \rho\mathbf{u} \cdot \nabla \frac{E}{\rho} + \nabla \cdot (p\mathbf{u})$$

$$+ \left[\frac{E}{\rho}\frac{\partial\rho}{\partial t} + \frac{E}{\rho}\nabla \cdot (\rho\mathbf{u})\right]$$

$$\frac{\partial E}{\partial t} = -\nabla \cdot (p\mathbf{u} + E\mathbf{u}) - P_0\Delta - \nabla \cdot \mathbf{q} + \mathbf{u} \cdot \mathbf{f}_v + \phi_\eta \qquad (3c\text{-}64)$$

The acoustic energy-flux vector can be identified as $p\mathbf{u} = \mathbf{J}$, inasmuch as this term represents the instantaneous rate at which one portion of the medium does mechanical work on a contiguous portion in the process of forwarding the sound energy. The time average of the sound-energy flux through unit area normal to $\mathbf{u}$ is defined as the *sound intensity*, $\langle \mathbf{J} \rangle \equiv \mathbf{I}$. Ordinarily it is only the time average of each term of (3c-64) that is of interest, but the equation itself holds at every instant and asserts that growth of the total energy density of a volume element is accounted for by the influx of acoustic and thermal energy across the boundaries of the element, by the energy dissipated in viscous losses, and by the work done by the equilibrium pressure on the volume element during condensation. The latter component is represented by $(-P_0\Delta)$ and by a corresponding linear term contained implicitly in E [cf. (3c-19)]. It is omitted in most textbook descriptions of acoustic energy density, the neglect being justified if at all on the grounds that the stored energy varies linearly with the dilatation and hence will have a vanishing net value when averaged over an integral number of periods or wavelengths, or over the entire region occupied by the sound field. Care must be taken to ensure that it does indeed vanish rigorously on the average inasmuch as the peak values of this component of energy storage are larger than the acoustic energy in the ratio P_0/p.

Acoustic Radiation Pressure. The appearance of the product $E\mathbf{u}$ as an additive term in the first right-hand member of (3c-64) is notable and represents the net energy density carried across the boundary of a volume element by convection, the net flow being measured by the divergence of the particle velocity.[1] No approximations have been made in deducing (3c-64), which holds, therefore, within the scope of validity of the basic assumptions.

It is significant to remark the fact that E is directly additive to p when the divergence term is written as $\nabla \cdot (p + E)\mathbf{u}$, thereby identifying the additive term as a *radiation pressure* whose magnitude at every instant is just equal to the total energy density, $E = \frac{1}{2}\rho\mathbf{u} \cdot \mathbf{u} + \rho\epsilon$. This interpretation can be fortified by revising (3c-64) by expanding $\nabla \cdot (E\mathbf{u}) = E(\nabla \cdot \mathbf{u}) + \mathbf{u} \cdot \nabla E$. The last term can be used to restore the material time derivative of E and the other can be merged with the linear term in P_0, yielding a revised power equation in the form

$$\frac{DE}{Dt} = -\nabla \cdot (p\mathbf{u}) - (P_0 + E)\Delta - \nabla \cdot \mathbf{q} + \phi_\eta + \mathbf{u} \cdot \mathbf{f}_v \qquad \text{(3c-65)}$$

The role of E as an additive, or radiation pressure is thus retained in (3c-65) where its time-independent part is now exhibited appropriately as a slight change in the equilibrium pressure.

When seeking to evaluate the net mechanical force due to radiation pressure on a material obstacle or screen exposed to a sound field, care must be taken to specify the boundary conditions and to account for *all* the reaction forces involved, including the steady-state interaction of the obstacle with the medium as well as the dynamic interaction of the obstacle with the sound field itself. Thus, for example, if a long tube is "filled" with a progressive plane wave, the walls of the tube, which interact only with the medium, would experience only the mean increment of the equilibrium pressure [cf. (3c-53)], and this would disappear if the walls were permeable to the medium, but not to the sound wave (e.g., with capillary holes). On the other hand, if a sound-absorbing screen were freely suspended athwart the wavefronts, it would experience just the pressure E shown by (3c-64) to be additive to p; but if the screen were to form an impermeable termination of the tube it would experience both components of pressure, including changes due to the enhancement of $\langle E \rangle$ by the reflected wave.[2]

3c-9. Sound Absorption and Dispersion. The basic manifestation of the absorption or attenuation of sound is the conversion of organized systematic motions of the particles of the medium into the uncoordinated random motions of thermal agitation.

[1] Schock, *Acustica* **3**, 181–184 (1953).

[2] The literature on radiation pressure is extensive, and much of it is confusing. The fundamentals are soundly discussed by L. Brillouin, "Les Tenseurs en mécanique et en élasticité," Dover Publications, New York, 1946. The influence of oblique incidence and of the reflection coefficient of the obstacle is discussed in detail by F. E. Borgnis, On the Forces upon Plane Obstacles Produced by Acoustic Radiation, *J. Madras Inst. Technology* **1** (2), pp. 171–210 (November, 1953); (3), pp. 1–33 (September, 1954); also a condensed version in *Revs. Modern Phys.* **25**, 653–664 (1953). A suggestive review, with a critical bibliography, has been given recently by E. J. Post, *J. Acoust. Soc. Am.* **25**, 55–60 (1953).

Various agencies of conversion can be identified as viscosity, heat conduction, or as some other mechanism that gives rise to a delay in the establishment of thermodynamic equilibrium; but all are mechanisms of interaction that lead to the same result, viz. that the energy of mass motion imparted intermittently to the medium by the sound source becomes increasingly disordered and "unavailable." Describing this in terms of the irreversible production of entropy leads to the definition of dissipation functions and paves the way for formulating an acoustic energy balance.

Equation of Continuity for Acoustic Energy. This may take the form of a statement that the mean net influx of sound energy across the boundaries of a volume element situated in a sound field must just balance the average time rate at which this energy is degraded, or made unavailable, throughout the volume element by irreversible increase of entropy; thus, by extension of (3c-20),

$$-\int_A J_i\,da_i = \int_V \frac{DE_{\text{diss}}}{Dt}\,dV = \int_V T\,\frac{DS_{\text{irr}}}{Dt}\,dV = \int_V (\phi_\kappa + \phi_\eta)\,dV \qquad \text{(3c-66)}$$

where the sound energy flux vector is $J_i = pu_i$, and E_{diss} is the degraded component of internal energy associated with the irreversible entropy S_{irr}.

The differential form of (3c-66) can be obtained in the usual way by using the divergence theorem to convert the surface integral to a volume integral. Then, after introducing the explicit forms of the dissipation functions, (3c-24*a*) and (3c-24*b*), the acoustic energy continuity relation becomes

$$\begin{aligned}-\nabla\cdot\mathbf{J} = -\frac{\partial(pu_i)}{\partial x_i} = \phi_\kappa + \phi_\eta = \frac{\kappa}{T}\left(\frac{\partial T}{\partial x_i}\right)^2 &+ \eta'\,\frac{\partial u_k}{\partial x_k}\frac{\partial u_i}{\partial x_i}\\ &+ \frac{1}{2}\eta\left[\left(\frac{\partial u_i}{\partial x_j}\right)^2 + \left(\frac{\partial u_j}{\partial x_i}\right)^2 + 2\,\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}\right] \qquad \text{(3c-67a)}\end{aligned}$$

where it is understood that only the time-independent parts of each side of (3c-67*a*) are to be retained. The algebraic complexity of dealing with (3c-67*a*) is considerably abated by considering only plane waves, for which case the running subscripts each reduce to unity and can be dropped. The plane-wave form of the acoustic-energy relation then becomes, after introducing P as an implicit variable in ∇T,

$$-\frac{\partial(pu)}{\partial x} = \frac{\kappa}{T}\left(\frac{DT}{DP}\right)^2\left(\frac{\partial p}{\partial x}\right)^2 + \eta\mathcal{V}\left(\frac{\partial u}{\partial x}\right)^2 \qquad \text{(3c-67b)}$$

in which $\eta\mathcal{V}$ has been written for $\eta' + 2\eta$ [cf. (3c-10)]. The thermal dissipation term can then be maneuvered into more suggestive form by further manipulation involving the equation of state $T = T(P,\rho)$ and various thermodynamic identities including the useful relation that holds for all fluids, $T\beta^2c^2 = C_p(\gamma - 1)$. This leads, still without approximation, and with the time average explicitly indicated, to

$$\left\langle -\frac{\partial(pu)}{\partial x}\right\rangle = \left\langle \eta\mathcal{V}\left(\frac{\partial u}{\partial x}\right)^2\right\rangle + \left\langle \frac{\kappa}{\rho C_p}\,\frac{[(\rho c^2/K_T) - 1]^2}{(\gamma - 1)\rho c^2}\left(\frac{\partial p}{\partial x}\right)^2\right\rangle \qquad \text{(3c-68)}$$

It can now be observed that p, u, and their derivatives must be known throughout the sound field in order to evaluate the sound energy flux and the dissipation functions that make up (3c-67*a*) or its reduced form (3c-68). On the other hand, if these field variables are known explicitly, the effects of dissipation will already be in evidence without recourse to (3c-68). Such a continuity equation for *acoustic* energy is therefore redundant, as might have been expected inasmuch as the conservation of energy has already been incorporated in the basic equations (3c-5), (3c-15), and (3c-23). Nevertheless, (3c-68) retains some logical utility as an auxiliary relation, even though it no longer needs to be relied on for the pursuit of absorption measures, at least for plane waves.

Exact Solution of the First-order Equations. An exact solution of the complete first-order equations (3c-26*a*), (3c-26*b*), (3c-26*c*) for the plane-wave case, and a definitive discussion of its implications, have been given recently by Truesdell.[1] The specific problem considered is that of forced plane damped waves in a viscous, conducting fluid medium. It is assumed that each of the first-order incremental state and field variables can be described by the real parts of

$$u_1 = u_{10} e^{j\omega t} e^{-(\alpha + jk)x} \tag{3c-69}$$

and of similar equations for ρ_1, p_1, θ_1. It is assumed that $(u_1)_{x=0} = u_{10}e^{j\omega t}$ is the simple-harmonic velocity imparted to the medium by the vibrating surface of a source located at $x = 0$, but the other amplitude coefficients may be complex in order to embody the phase angles by which these variables lead or lag u_1. The exponent expressing time dependence is written $+j\omega t$, as required in order to preserve both the conventional form $R + jX$ for complex impedances *and* the positive sign for inductive or mass reactance. The attenuation constant α and the phase constant $k \equiv \omega/c$, or $k_0 \equiv \omega/c_0$, are the real and imaginary parts of the complex propagation constant $\chi \equiv \alpha + jk$; and $c_0 \equiv (\partial P/\partial \rho)_s^{\frac{1}{2}}$ is the *reference* value of sound speed.

When the assumed solutions (3c-69) are systematically introduced in (3c-26*a*), (3c-26*b*), and (3c-26*c*), three algebraic equations in ρ_1, u_1, θ_1 are obtained, as follows:

$$\begin{aligned}
&\rho_0(\alpha + jk)u_1 && -j\omega\rho_1 && = 0\\
&[j\omega\rho_0 - \eta\mathcal{V}(\alpha + jk)^2]u_1 && -(\alpha + jk)\left[\frac{c_0^2}{\gamma}(\rho_1 + \beta_0\rho_0\theta_1)\right] && = 0\\
&-\frac{\gamma - 1}{\beta_0}(\alpha + jk)u_1 && +\left[j\omega - \frac{\kappa}{\rho_0 C_v}(\alpha + jk)^2 + \mathfrak{q}\right]\theta_1 && = 0
\end{aligned} \tag{3c-70}$$

If these equations are indeed to admit solutions of the assumed form (3c-69), the determinant of the coefficients of u_1, ρ_1, and θ_1 must vanish. The characteristic or *secular equation* formed in this way (Kirchhoff, for perfect gases, 1868; extended to any fluid with arbitrary equation of state by P. Langevin[2]) turns out to be a biquadratic in the dimensionless complex propagation variable $(\alpha + jk)/k_0$. Writing this out in full, however, will be facilitated by first considering the question of how best to specify the properties of the medium.

Dimensional Analysis and Absorption Measure. Examination of (3c-70) reveals that, in addition to $(\alpha + jk)/k_0$ and the three independent variables, there are 10 parameters that pertain to the behavior of the medium at the angular frequency ω. One of these could be eliminated, in principle at least, by using the relation $T\beta^2c^2 = (\gamma - 1)C_p$, leaving 9 that are independent: C_p, C_v, η, η', κ, ρ_0, c_0, $\mathfrak{q}$, and ω. Then, since each of these can be expressed in terms of 4 basic dimensional units (e.g., mass, length, time, and temperature), it follows from the pi theorem of dimensional analysis[3] that just 5 independent dimensionless ratios can be formed out of combinations of these 9 parameters. This leads to a functional expression of the absorption measure in the symbolic form

$$\frac{\alpha + jk}{k_0} = \psi\left(\frac{C_p}{C_v}, \frac{\eta'}{\eta}, \frac{\eta C_p}{\kappa}, \frac{\omega\eta}{\rho_0 c_0^2}, \frac{\mathfrak{q}}{\omega}\right) \tag{3c-71}$$

The first two ratios have already been incorporated in γ and the viscosity number $\mathcal{V} \equiv 2 + \eta'/\eta$; the third is the Prandtl number $\mathcal{P} \equiv \eta C_p/\kappa$, and the fourth and fifth can be identified as Stokes numbers $S \equiv \omega\eta/\rho_0 c_0^2$ and $S' \equiv \omega/\mathfrak{q}$. The present purpose

[1] C. A. Truesdell, Precise Theory of the Absorption and Dispersion of Forced Plane Infinitesimal Waves According to the Navier-Stokes Equations, *J. Rational Mechanics and Analysis* **2**, 643–741 (October, 1953).

[2] Reported by Biquard, *Ann. phys.* (11) **6**, 195–304 (1936).

[3] E. Buckingham, *Phys. Rev.* **4**, 345 (1914); *Phil. Mag.* (6) **42**, 696 (1921).

is served somewhat better by substituting for the third and fourth ratios their products with the dimensionless viscosity number, thus defining a frequency number X and thermoviscous number Y through

$$X \equiv \mathcal{V}S = \frac{\omega\eta\mathcal{V}}{\rho_0 c_0^2} \qquad Y \equiv (\mathcal{P}\mathcal{V})^{-1} = \frac{\kappa}{\eta\mathcal{V}C_p} \qquad XY = \frac{\omega\kappa}{\rho_0 c_0^2} \qquad \text{(3c-72)}$$

The frequency parameter X also provides a natural criterion for designating frequencies as "low," "medium," or "high" according to whether X is much less than, comparable with, or much greater than unity. It may also be noted that, for nearly perfect gases, $\rho_0 c_0^2 \doteq \gamma P_0$, from which it follows that $X_{\text{gas}} \doteq (\omega/P_0)(\eta\mathcal{V}/\gamma)$. Hence variation of pressure may be used to extend in effect the accessible range of frequency in measurements on gases, and the ratio ω/P_0 is a proper parameter in terms of which to report such results.

Solutions of the Characteristic Equation. If the dimensionless ratios discussed above are now introduced in the expanded determinant of the coefficients of (3c-70), the resulting Kirchhoff-Langevin secular equation can be written as

$$\left(1 - \frac{j}{S'}\right) + \left(\frac{\alpha + jk}{k_0}\right)^2 \left[1 + jX(1 + \gamma Y) + \frac{\gamma X - j}{\gamma S'}\right] + \left(\frac{\alpha + jk}{k_0}\right)^4 XY(j - \gamma X) = 0 \qquad \text{(3c-73)}$$

The standard "quadratic formula" can be used at once to solve (3c-73) for the reciprocal square of the propagation constant;

$$-2\left(1 - \frac{j}{S'}\right)\left(\frac{k_0}{\alpha + jk}\right)^2 = 1 + \frac{X}{S'} + j\left[X(1 + \gamma Y) - \frac{1}{\gamma S'}\right] \pm \left[\left(1 + \frac{X}{S'}\right)^2 - \left[X(1 - \gamma Y) - \frac{1}{\gamma S'}\right]^2 + 2j\left\{X[1 - (2 - \gamma)Y] + X^2\frac{1 - \gamma Y}{S'} - \frac{[1 + (X/S')]}{\gamma S'}\right\}\right]^{\frac{1}{2}} \qquad \text{(3c-74a)}$$

Skillful abbreviation might allow this complete solution to be carried somewhat further but no algebraic magic can lighten very much the burden of depicting the behavior of α and k as a function of *four* independent parameters—and it might have been five but for the welcome fact that $\mathcal{V}$ does not appear except as embodied in X and Y. Moreover, each parameter that does appear in (3c-74*a*) occurs in one or more product combinations, and hence it can *not* be assumed in general that the effects of viscosity and heat exchange will be linearly additive. The common practice of assessing these one at a time and then superimposing the results must therefore be considered unreliable unless justified explicitly and quantitatively. Nevertheless, something must give, and it is customary to abandon first the radiant-heat exchange, at least temporarily, by letting S' become infinite in (3c-74*a*). With this simplification, and with some new abbreviations, (3c-74*a*) becomes

$$-2\left(\frac{k_0}{\alpha + jk}\right)^2 = 1 + jX(1 + \gamma Y) \pm \{1 - X^2(1 - \gamma Y)^2 + j2X[1 - (2 - \gamma)Y]\}^{\frac{1}{2}}$$
$$\equiv G + jH = 1 + jX(1 + \gamma Y) \pm (E + jF)^{\frac{1}{2}} \qquad \text{(3c-74b)}$$
$$E \equiv 1 - X^2(1 - \gamma Y)^2 \qquad F \equiv 2X[1 - (2 - \gamma)Y]$$

This equation has two pairs of noncoincident complex roots, but only the one of each pair that has a nonnegative real part corresponding to real attenuation is to be retained. These two physical solutions comprise the two branches of a complex square root; one branch pertains to typical compressional sound waves identified as type I, the other to so-called thermal waves identified as type II. It is an unwarranted oversimplification, however, to describe these simply as "pressure" waves and "thermal" waves inasmuch as *all* the state and condition variables—pressure, density, velocity, temperature, heat flux, etc.—are simultaneously entrained and propagated by *each* wave-type, and waves of *both* types are always excited simultaneously by any source. On the other hand, the absorption and dispersion measures for waves of type I and type II will, in general, be quite different and will vary differently with the frequency parameter X and with the thermoviscous parameters γ and Y that characterize the fluid. For example, type II waves are so rapidly attenuated in ordinary fluids at accessible frequencies that they cannot be observed, whereas in strongly conducting liquids such as mercury (and perhaps in liquid helium II) the absorption for type II waves becomes less than for type I waves when the frequency is high enough for X to exceed $\frac{1}{3}$.

It should be noticed, parenthetically, that if the basic first-order equations (3c-70) had not been restricted to plane waves, the last term of (3c-26*b*) would not have dropped out. Instead, there would have turned up eventually in (3c-70) a pair of terms in the first-order vector velocity potential $\mathbf{A}_1$ [see (3c-55)] on the basis of which it would have been predicted that still another type of allowed wave motion can exist in viscous fluids—a transverse *viscous wave* that is propagated by virtue of the transverse shear reactions due to viscosity.[1]

Viscothermal Absorption and Dispersion Measures. The problem of branch determination arising in the solution of (3c-64*b*) has been discussed thoroughly by Truesdell.[2] One view of it can be expressed by writing the formal solution in the explicit form

$$\frac{\alpha}{k} \equiv \frac{A}{2\pi} = \frac{H}{+(G^2 + H^2)^{\frac{1}{2}} + G} \qquad \left(\frac{c}{c_0}\right)^2 = \frac{2(G^2 + H^2)}{+(G^2 + H^2)^{\frac{1}{2}} + G}$$
$$2G = 1 \pm f(h)(+E^{\frac{1}{2}}) \qquad 2H = X(1 + \gamma Y) \pm (\operatorname{sgn} F)g(h)(+E^{\frac{1}{2}}) \tag{3c-75a}$$

(upper signs yield type I waves, lower signs type II waves)

$$h \equiv \frac{F}{E} \qquad \begin{aligned} f(h) &\equiv +\sqrt{2}\,[+(1 + h^2)^{\frac{1}{2}} + 1] = +\cosh \tfrac{1}{2}(\sinh^{-1} h) \\ g(h) &\equiv +\sqrt{2}\,[+(1 + h^2)^{\frac{1}{2}} - 1] = +\sinh \tfrac{1}{2}(\sinh^{-1} h) \end{aligned} \tag{3c-75b}$$

where the plus signs associated with roots denoted by fractional exponents indicate that the principal or positive root is to be used. The solution (3c-75*a*) can now be attacked frontally, either by means of power-series expansions for large or small values of X, or by resorting to brute-force numerical computation for intermediate frequencies. The several square-root operations on complex quantities required by the latter procedure are often facilitated by using the f and g functions defined by (3c-75*b*), for which the principal values have been tabulated.[3]

The clue to a basis for classifying fluids according to their viscothermal behavior is afforded by noting that the algebraic sign of F appears in (3c-75*a*) in such a way as to interchange the wave types when F changes sign, and that this occurs when $(2 - \gamma)Y$ passes through unity. On this basis, one may categorize fluids as *strong* conductors if Y is greater than $(2 - \gamma)^{-1}$. The contrary alternative can be further subdivided usefully[2] into *weak* conductors for which Y is less than γ^{-1}, and *moderate* conductors for which Y has intermediate values. Most liquids (including the liquefied noble gases) qualify as weak conductors, most gases as moderate conductors. On the other hand, the fact that mercury, the molten metals, and liquid helium II rank as strong

[1] Rayleigh, "Theory of Sound," vol. II, §§347; Mason, *Trans. ASME* **69**, 359–367 (1947); Epstein and Carhart, *J. Acoust. Soc. Am.* **25**, 553–565, [557] (1953).

[2] C. A. Truesdell, Precise Theory of the Absorption and Dispersion of Forced Plane Infinitesimal Waves According to the Navier-Stokes Equations, *J. Rational Mechanics and Analysis* **2**, 643–741 (October, 1953).

[3] G. W. Pierce, *Proc. Am. Acad. Arts Sci.* **57**, 175–191 (1922).

conductors emphasizes the value of including a wide range of parameter values in any general survey of thermoviscous behavior.

For weak or moderate conductors, the absorption and dispersion measures for type I waves at moderately low frequencies can be expressed with any desired precision by means of power-series expansions in the frequency number X:

$$\left(\frac{c}{c_0}\right)^2 = 1 + \frac{1}{4}X^2[3 + 10(\gamma - 1)Y - (\gamma - 1)(7 - 3\gamma)Y^2] + 0(X^4)$$

$$\frac{\alpha}{k_0} \equiv \frac{A_0}{2\pi} = \frac{1}{2}X\left\{1 + (\gamma - 1)Y - \frac{1}{8}X^2[5 + 35(\gamma - 1)Y + (\gamma - 1)(35\gamma - 63)Y^2 + (\gamma - 1)(5\gamma^2 - 30\gamma + 33)Y^3]\right\} + O(X^5) \quad (3c\text{-}76)$$

$$\frac{\alpha}{k} \equiv \frac{A}{2\pi} = \frac{1}{2}X\left\{1 + (\gamma - 1)Y - \frac{1}{4}X^2[1 + 11(\gamma - 1)Y - (\gamma - 1)(23 - 11\gamma)Y^2 + (\gamma - 1)(\gamma^2 - 10\gamma + 13)Y^3]\right\} + O(X^5)$$

Note that $\alpha/k \equiv \alpha\lambda/2\pi \equiv A/2\pi$, where A is the amplitude attenuation per wavelength, and that α/k_0 is similarly related to the attenuation per reference wavelength λ_0. The series (3c-76) can be used with confidence for almost any values of γ and Y so long as the frequency is low enough to keep $X < 0.1$, and for a somewhat wider range of X when certain restrictions on γ and Y are satisfied.[1]

On the other hand, for frequencies high enough to make $X^{-2} \ll 1$, the absorption and dispersion are given, within $O(X^{-2})$, by

$$\frac{(c/c_0)^2}{2X} = \frac{\alpha}{k} \equiv \frac{A}{2\pi} = \frac{A_0{}^2X}{2\pi^2} = \left(\frac{\alpha}{k_0}\right)^2 2X = 1 - \frac{1 - Y}{(1 - \gamma Y)X} \quad (3c\text{-}77)$$

It can be inferred at once from (3c-77) that, for sufficiently high frequencies, dispersion is always anomalous (i.e., speed *increases* with frequency) regardless of γ and Y; that $\alpha/k = A/2\pi$ approaches the limit 1, and that α/k_0 and A_0 recede to zero as the actual wavelength decreases with respect to the reference wavelength λ_0. It also follows, from comparison of this result with (3c-76_3), that as frequency increases, $\alpha = A/\lambda = A_0/\lambda_0$ *will always have at least one maximum that is characteristic of viscothermal resonance.* The frequency at which this resonance occurs lies in the range $X = 1$ to 1.7, but the peak is relatively broad and flat and often cannot be located experimentally with high precision.

It can also be deduced from (3c-77) that the asymptotic speed of sound at very high frequencies will always be determined by viscosity alone, without regard for the form of the equation of state; thus,

$$(c^2)_{X\to\infty} = \frac{2\omega\eta\mathcal{V}}{\rho} \quad (3c\text{-}78a)$$

Under the same limiting conditions, the asymptotic speed of type II, or "thermal," waves is similarly determined by thermal conductivity alone, according to

$$(c'^2)_{X\to\infty} = \frac{2\omega\kappa}{\rho_0 C_p} \quad (3c\text{-}78b)$$

The steady increase of c' with $\omega^{\frac{1}{2}}$ predicted by (3c-78b) has sometimes been cited as a basis for denying that second sound in helium II, which displays small dispersion and low attenuation,[2] can be a type II thermal wave of the sort predicted by viscothermal

[1] Truesdell, *J. Rational Mechanics and Analysis* **2**, 643–741 (October, 1953).

[2] Peshkof, *J. Phys.* (*U.S.S.R.*) **8**, 381 (1944); **10**, 389–398 (1946); Lane, Fairbank, and Fairbank. *Phys. Rev.* **71**, 600–605 (1947).

theory. This conclusion is probably correct but the argument is faulty inasmuch as the vanishing viscosity of the superfluid would make it more appropriate to use as a type criterion the behavior predicted for the limiting condition $X \to 0$. Thus, if the Kirchhoff-Langevin secular equation (3c-73) is reduced by letting $X \to 0$ while XY is held fixed, and if XY is then allowed to increase indefinitely as required by the superconductivity of helium II, what is left of (3c-73) *does* have a pair of roots for which the attenuation vanishes and the speed is nondispersive, viz., $\alpha = A_0 = 0$, and $c = c_0/\gamma^{\frac{1}{2}}$. This result looks, at first sight, like just an isothermal velocity for type I waves, as might be expected to prevail if uniform temperature were enforced by infinite conductivity. On the other hand, the wave types would be expected to interchange, according to (3c-75*a*), as Y becomes very large; and one has also to deal with the standing conclusion that any viscosity however small will eventually take over control of dispersion when X departs sufficiently from zero. These remarks are intended to emphasize primarily the fact that the problem of branch determination, or type identification, under such extreme circumstances needs probably to be attacked by considering the relative *rates* at which the various limiting conditions are approached. Other considerations need also to be taken into account, of course, in dealing with the two-fluid-mixture theory of liquid helium; but it seems clear that further inquiry is warranted concerning the relevance of classical viscothermal concepts now that a more exact theory of these effects is available.

The Kirchhoff approximation for weak or moderate conductors at low frequencies can be obtained directly from (3c-76) by neglecting terms in X^2 or higher. The dispersion is thereby predicted to be negligible, so that $c \doteq c_0$; and the "Kirchhoff" attenuation α_K is given by

$$\alpha_K = \frac{1}{2} k_0[X + (\gamma - 1)XY] = \frac{1}{2} k_0 S \left(\mathcal{V} + \frac{\gamma - 1}{\mathcal{P}} \right)$$
$$= \frac{\omega^2}{2\rho_0 c_0^3} \left[\eta \mathcal{V} + \frac{(\gamma - 1)\kappa}{C_p} \right] \tag{3c-79a}$$

If the Stokes relation is then presumed, by setting $\mathcal{V} = \frac{4}{3}$ (which neither Kirchhoff nor Stokes himself did in this connection), (3c-79*a*) becomes

$$\alpha_C = \frac{1}{2} k_0 S \left(\frac{4}{3} + \frac{\gamma - 1}{\mathcal{P}} \right) = \frac{\omega^2}{2\rho_0 c_0^3} \left[\frac{4}{3} \eta + \frac{(\gamma - 1)\kappa}{C_p} \right] \tag{3c-79b}$$

The absorption predicted by (3c-79*b*) is commonly, but not very appropriately, referred to as "classical"; but such an emasculated theoretical prediction neither accounts adequately for the attenuation observed experimentally, except in the case of a few monatomic gases, nor does it do justice to the essential content of the classical theory of viscous conducting fluids.

Even when terms through X^2 are included, no change occurs in the odd function α/k_0, but dispersion is then predicted according to (3c-76$_1$) which accounts for the second-order effects of both compressional and shear viscosity, heat conduction, and their interaction. This dispersion is anomalous for weak or moderate conductors (small Y) but becomes normal if the speed-reducing influence of thermal conductivity becomes large enough to make $(7 - 3\gamma)Y > 10$. On the other hand, if heat exchange were to be ignored altogether, the first two terms of (3c-76$_1$) would give, for the dispersion due to viscosity alone,

$$\left(\frac{c}{c_0}\right)^2 \doteq 1 + \frac{3}{4} X^2 = 1 + \frac{3}{4} \left(\frac{\omega \eta \mathcal{V}}{\rho_0 c_0^2}\right)^2$$
$$c \doteq c_0 \left[1 + \frac{3}{8} \left(\frac{\omega \eta \mathcal{V}}{\rho_0 c_0^2}\right)^2 \right] \tag{3c-80}$$

Absorption and Dispersion Due to Heat Radiation. The effects of heat exchange by radiation, which were abandoned above in order to make (3c-74) more manageable, can now be assessed by reverting to (3c-73). The nonlinear interaction between radiation and viscosity will be neglected, for the sake of expediency, even though (3c-74) suggests that it may be as large as second order. The primary effects of viscosity and heat conduction can be eliminated from (3c-73) by letting both X and XY go to zero while holding the frequency variable $S' = \omega/\mathfrak{q}$ finite. This reduces the characteristic secular equation to the simple quadratic form

$$\gamma(S' - j) + \left(\frac{\alpha + jk}{k_0}\right)^2 (\gamma S' - j) = 0 \tag{3c-81}$$

which can be solved directly to yield the following exact expressions for the attenuation and dispersion due to radiation alone:

$$\begin{aligned} \frac{A}{2\pi}\left(\frac{c_0}{c}\right)^2 &= \frac{\alpha}{k}\left(\frac{c_0}{c}\right)^2 = \gamma S' \frac{\gamma - 1}{2[1 + (\gamma S')^2]} \\ \left(\frac{A_0}{2\pi}\right)^2 &= \left(\frac{\alpha}{k_0}\right)^2 = \frac{1}{2}\gamma \frac{(1 + S')^{\frac{1}{2}}(1 + \gamma^2 S'^2)^{\frac{1}{2}} - (1 + \gamma S'^2)}{1 + (\gamma S')^2} \\ \left(\frac{c}{c_0}\right)^2 &= \frac{2}{\gamma} \frac{1 + (\gamma S')^2}{(1 + \gamma S'^2) + (1 + S'^2)^{\frac{1}{2}}(1 + \gamma^2 S'^2)^{\frac{1}{2}}} \end{aligned} \tag{3c-82}$$

These equations indicate that both attenuation and dispersion become vanishingly small for either very large or very small values of S', and that a maximum of attenuation occurs in mid-range, near the single point of inflection of the dispersion curve. This absorption peak is characterized by

$$\begin{aligned} \left(\frac{\alpha}{k}\right)_{\max} &= \frac{\gamma^{\frac{1}{4}} - 1}{\gamma^{\frac{1}{2}} + 1} \qquad S'_{\max A} = \gamma^{-\frac{1}{2}} \qquad \tau_{\text{rad}} = \frac{2\pi\gamma^{\frac{1}{2}}}{\mathfrak{q}} \\ \left(\frac{\alpha}{k_0}\right)_{\max} &= \frac{A_0}{2\pi} = \frac{\gamma - 1}{[8(\gamma + 1)]^{\frac{1}{2}}} \qquad S'_{\max A_0} = \gamma^{-1}\frac{(3\gamma + 1)^{\frac{1}{2}}}{(\gamma + 3)^{\frac{1}{2}}} \end{aligned} \tag{3c-83}$$

There is a curious dearth of quantitative information concerning the radiation coefficient $\mathfrak{q}$, and little is added to this by noticing the low attenuation and negligible dispersion observed for a wide range of audible sounds in air since these might correspond to values of S' either far above or far below the resonance peak described by (3c-83). The choice $S' \gg 1$ is unambiguously dictated, however, by the fact that the observed speed of sound is very close to the isentropic value c_0, whereas (3c-82_5) indicates that the isothermal speed $c_0/\gamma^{\frac{1}{2}}$ would prevail if $\mathfrak{q}$ were large enough to make S' small for all audio frequencies. Truesdell[1] has pointed out that these conclusions leave still in effect a prediction that at some lower subaudible frequency a peak of attenuation should appear with a magnitude $A_0 = 0.185\pi$ (≈ 5 db per reference wavelength). This absorption peak has not been observed yet, at least deliberately, although its possible bearing on the acoustical character of thunder might be worth investigating.

Relaxation Processes and Sound Absorption. The foregoing analysis of heat exchange by radiation puts in evidence the first example of what would now be called a typical relaxation process. The characteristic feature of such a process, in so far as the gross hydrodynamical response of the medium is concerned, is the existence of two relations among the state variables, one of which prevails asymptotically for slow variations, the other for rapid changes. Such bivalent behavior is typical of fluid mixtures containing two interacting components, such as a partly dissociated gas[2] or an ionic solution.[3] In these cases the relative concentrations of the two components either follow faithfully, in quasi-static equilibrium, the dictates of slowly changing external variables, or else, at the other asymptotic limit, they do not change at all

[1] C. A. Truesdell, *J. Rational Mechanics and Analysis* **2,** 643–741 [666] (October, 1953).
[2] Einstein, *Sitzber. deut. Akad. Wiss. Berlin Math.-Phys. Kl.* **1920,** 380–385.
[3] Liebermann, *Phys. Rev.* **76,** 1520–1524 (1949).

when the finite reaction rate is such that the external variables can complete cyclic changes too rapidly for the concentrations to "follow." A different but comparable kind of mixture is exemplified by an ensemble of atoms or molecules capable of being excited to different energy levels, of which the most common example is a diatomic gas in which the rotational degrees of freedom may or may not share the cyclic work of compression depending on whether an appropriately normalized frequency variable is "low" or "high."

The *physical* problem of characterizing the rate-dependent properties of mixtures can be studied without regard for its acoustical consequences, and various approaches to this problem have turned on the assignment of two or more different internal or "partial" temperatures, different compressibilities, specific heats, etc. All the physical theories of pure relaxation appear to converge, however, in predicting the same *acoustical* behavior; viz., at low frequencies an asymptotic speed of sound c^0, a transition region of anomalous dispersion ($dc/d\omega > 0$) within which a maximum of attenuation occurs, and at high frequencies an asymptotic sound speed c^∞ which can be related to c^0 by writing $K \equiv c^0/c^\infty \leq 1$, where K is a material constant of the two-component medium. It follows then that, when the constant K and a dimensionless frequency variable X' can be properly identified and interpreted in terms of the *physical* mechanism involved, the *acoustical* behavior for any pure relaxation process will be described exactly by the following expressions derived from (3c-82) and (3c-83) by substitution:

$$\begin{aligned}
\left(\frac{c}{c^0}\right)^2 &= \frac{2(1 + X'^2)}{1 + K^2X'^2 + [(1 + K^4X'^2)(1 + X'^2)]^{\frac{1}{2}}} \\
&\doteq \frac{1 + X'^2}{1 + K^2X'^2} \\
\frac{\alpha}{k}\left(\frac{c^0}{c}\right)^2 &= \frac{1}{2}\frac{(1 - K^2)X'}{1 + X'^2} \\
\left(\frac{\alpha}{k}\right)_{\max} &= \frac{1 - K}{1 + K} \qquad \left(\frac{\alpha}{k_0}\right)_{\max} = \frac{1 - K^2}{[8(1 + K^2)]^{\frac{1}{2}}} \\
X'_{\max A} &= K^{-1} = \frac{c^\infty}{c^0} \qquad X'_{\max A_0} = \left(\frac{3 + K^2}{1 + 3K^2}\right)^{\frac{1}{2}}
\end{aligned} \tag{3c-84}$$

These equations revert exactly to (3c-82) and (3c-83) when the substitutions $K^2 = \gamma^{-1}$, and $X' = \gamma S'$, are made, and when a factor γ^{-1} is introduced to convert the low-frequency reference speed c^0 to the usual isentropic reference c_0.

The "resonance" frequency characterizing a relaxation process is usually defined as the angular frequency at which the maximum attenuation per wavelength, $A = \alpha\lambda$, occurs; thus, $\omega_r \equiv 2\pi/\tau_r = (\omega/X')X'_{\max A}$, where τ_r is the related "relaxation period." It has been pointed out that *any* mechanism of sound absorption can be interpreted as a relaxation phenomenon by suitably defining its relaxation time. For example, viscosity and heat-conduction "relaxation times" and their associated "resonance frequencies" can be defined by writing

$$\tau_v = \frac{2\pi}{\omega_v} = \frac{X}{\omega}\frac{4}{3\mathcal{V}} = \frac{\frac{4}{3}\eta}{\rho_0 c_0^2} \qquad \tau_\kappa = \frac{2\pi}{\omega_\kappa} = \frac{XY}{\omega} = \frac{\kappa}{\rho_0 c_0^2 C_p} \tag{3c-85}$$

Note that ω_v is specified in such a way that it reduces to ω/X when $\mathcal{V}$ has the Stokes-relation value $\frac{4}{3}$. When these relaxation frequencies are introduced in (3c-79) and (3c-80), the second-order dispersion and the Kirchhoff linear approximation for attenuation become

$$\begin{aligned}
c &\doteq c_0\left[1 + \frac{3}{8}(2\pi)^2\left(\frac{3\mathcal{V}}{4}\right)^2\frac{\omega^2}{\omega_v^2}\right] \\
\alpha_K &= \pi k_0\left[\frac{3\mathcal{V}}{4}\frac{\omega}{\omega_v} + (\gamma - 1)\frac{\omega}{\omega_\kappa}\right]
\end{aligned} \tag{3c-86}$$

When the fluid medium consists of an ideal monatomic gas, the physical significance of the relaxation times τ_v and τ_κ can readily be interpreted as the time required for subsidence of a momentary departure from the equilibrium distribution of energy among the translational degrees of freedom. In the classical kinetic theory of gases, this recovery time is shown to be approximately $L/\bar{v}$, the mean free path divided by the mean molecular velocity.[1] The conformity of the definitions (3c-85) with this concept can then be verified by recalling the kinetic-theory evaluations of viscosity $[\eta \doteq \frac{1}{2}\rho\bar{v}L]$, thermal conductivity $[(\kappa/C_p) \doteq (5/4\gamma)\rho\bar{v}L]$, and the speed of sound $[c \doteq 0.74\bar{v}]$. These considerations show, incidentally, that for such a gas the attenuation per reference wavelength is contributed almost equally by viscosity and heat conduction, and is proportional to the ratio of mean free path to wavelength.

The precise physical significance of τ_v and τ_κ is less obvious for polyatomic gases and liquids; but if this is glossed over, the frequency ratios $\frac{3}{2}\pi\mathcal{V}\omega/\omega_v$, $\frac{4}{3}\omega_v/\omega_\kappa\mathcal{V}$, and $2\pi\omega/\omega_\kappa$ can be substituted directly for X, Y, and XY in any of the viscothermal relations deduced above. Merely introducing these "relaxation" frequencies, however, does not invest heat conduction or viscosity with any new or different relaxation-like properties, and the exact viscothermal theory, in whatever symbols expressed, continues to predict that sound speed will increase indefinitely with frequency, that A_0 will display a typical broad maximum for some X in the range 1 to 1.7 (depending on the thermoviscous parameters γ and Y), that $(A_0)_{\max}$ will always have about the same magnitude $(\alpha/k_0 \approx \frac{1}{3})$, and that the peak in A_0 can be made to occur at any chosen actual frequency by suitable assignment of the viscosity number $\mathcal{V}$. [cf. (3c-72), (3c-85)]. In contrast with this behavior, a pure relaxation phenomenon would call for the sound speed to level off at the high-frequency limit given by K^{-1}, and would display a maximum in A_0 that increases in height and retreats toward higher frequencies as the speed increment $c^\infty - c^0$ increases and K varies from 1 toward zero.

Allusion has already been made to the established fact that measured values of attenuation usually exceed the "classical" prediction (3c-79*b*) and often exhibit one or more maxima at finite frequencies. As a matter of fact, even when the complete consequences of the classical theory are taken into account, and when the viscosity number is adjusted to make the predicted attenuation at low frequencies correspond with experiment, the classical viscothermal theory still fails to account for all the experimental facts, but *for a reason that is just the opposite of that usually advanced*, namely, because it then predicts *too much* attenuation at the resonance peak and at higher frequencies! In spite of this latent contradiction, the alleged failure of "classical" theory as represented by (3c-79*b*) (which is, after all, only *part* of an *approximate* solution of the *linearized first-order* equations) has stimulated widespread efforts to repair its deficiency by invoking a wide variety of relaxation and other theories,[2] many of which have been marred by an *ad hoc* flavor that renders them little more than examples of ingenuity in curve fitting.

Measurements of absorption and dispersion in rarefied helium gas over a wide range of the frequency variable S have confirmed in all essential details the pattern of behavior predicted by the *exact* viscothermal theory.[3] Unless the classical concepts of viscosity and heat conduction are to be abandoned altogether, therefore, logic demands that the *exact* viscothermal theory be accepted as the foundation on which to erect any more complete analysis of sound absorption in media less idealized than rarefied

[1] Jeans, "Dynamical Theory of Gases," 2d ed., pp. 260–262, Cambridge University Press, Cambridge, England, 1916.

[2] For reviews of what has been called the "exuberant literature" dealing with relaxation and other theories of sound absorption, see Kneser, *Ergeb. exakt. Naturwiss.* **22,** 121–185 (1949); Markham, Beyer, and Lindsay, *Revs. Modern Phys.* **23,** 353–411 (1951); Kittel, *Phys. Soc.* (*London*), *Repts. Progr. in Phys.* **11,** 205–247 (1948); see also, for background, W. T. Richards, *Revs. Modern Phys.* **11,** 36–64 (1939).

[3] Greenspan, *Phys. Rev.* **75,** 197–198 (1949); *J. Acoust. Soc. Am.* **22,** 568–571 (1950).

helium. A good many "honest" relaxation mechanisms do exist and must be accounted for, but in the accounting these effects should presumably be regarded as factors perturbing the fundamental thermoviscous behavior rather than the converse. The two-fluid-mixture theory of relaxation effects seems best adapted for inclusion in such a compound analysis, and a start in this direction has already been made.[1] Much remains to be done, however, before this basic acoustical problem can be said to be understood.

3c-10. Characteristic Acoustic Impedance of a Thermoviscous Medium. When the first-order sound pressure p_1 is put back into (3c-70$_2$) [by tracing its last term back through (3c-25$_{11}$)], this equation of motion can be rewritten at once in terms of the specific acoustic impedance, as follows:

$$\begin{aligned}
&[j\omega\rho_0 - (\alpha + jk)^2\eta\mathcal{V}]u_1 - (\alpha + jk)p_1 = 0 \\
\frac{p_1}{u_1} &\equiv Z = jk\rho_0 c(\alpha + jk)^{-1} - \eta\mathcal{V}(\alpha + jk) \\
&= \rho_0 c\left(1 - j\frac{\alpha}{k}\right)^{-1} - j\rho_0 c\,\frac{\omega\eta\mathcal{V}}{\rho_0 c_0^2}\left(\frac{c_0}{c}\right)^2\left(1 - j\frac{\alpha}{k}\right) \\
\frac{p_1}{\rho_0 c u_1} &\equiv z = \left(1 - j\frac{\alpha}{k}\right)^{-1} - jX\left(\frac{c_0}{c}\right)^2\left(1 - j\frac{\alpha}{k}\right)
\end{aligned} \qquad (3c\text{-}87)$$

The normalized specific impedance, or *specific impedance ratio*, $(p_1/\rho_0 c u_1) \equiv z$, which would be unity in the nondissipative case, is now in a form to be evaluated by direct substitution of the series expansions (3c-76). After some manipulation, and retaining only terms through X^2 and Y^2, the impedance ratio can be put in the form

$$\begin{aligned}
\frac{p_1}{\rho_0 c u_1} &= 1 - \frac{\alpha}{k}\left[\frac{\alpha}{k} + X\left(\frac{c_0}{c}\right)^2\right] + j\left[\frac{\alpha}{k} - X\left(\frac{c_0}{c}\right)^2\right] \\
&= 1 - \frac{1}{4}X^2[3 + 4(\gamma - 1)Y + (\gamma - 1)^2Y^2] + O(X^4) \\
&\qquad - j\left\{\frac{1}{2}X[1 - (\gamma - 1)Y] + O(X^3)\right\}
\end{aligned} \qquad (3c\text{-}88)$$

It follows that sound pressure *lags* the particle velocity when $(\gamma - 1)\kappa/\eta\mathcal{V}C_p$ is less than unity, as it is for the common fluids under ordinary conditions; but pressure *leads* the particle velocity when the ratio of heat conductivity to viscosity is high enough to make $(\gamma - 1)\kappa > \eta\mathcal{V}C_p$.

3c-11. Thermal Noise in the Acoustic Medium. The mode of motion that is heat furnishes a restless background of noise that underlies all acoustical phenomena. The magnitude and nature of this thermal noise can be assessed by appealing to concepts drawn from such apparently unrelated sources as architectural acoustics, elementary quantum theory, and the classical kinetic theory of gases.

The scheme of analysis can be described simply: the thermoacoustic noise energy density, as measured by the mean-square sound pressure, is set equal to the density of the internal energy of thermal agitation associated with the translational degrees of freedom of the molecules composing the medium. It is then postulated that these molecular motions of thermal agitation can be regarded as a vector summation of the motions associated with a three-dimensional manifold of compressional standing waves, each behaving as it would in an ideal continuous medium having the same gross mechanical and elastic properties that characterize the actual medium. Each of these standing-wave systems thus constitutes an allowed, thermally excited, normal mode of vibration, or degree of freedom, to which can be assigned, in accordance with elementary quantum theory, the average energy

[1] Z. Sakadi, *Proc. Phys.-Math. Soc. Japan* (3) **23**, 208–213 (1941); Meixner, *Acustica* **2**, 101–109 (1952).

$$\frac{\text{Energy}}{\text{Mode}} = \frac{hf}{\exp(hf/kT) - 1} \tag{3c-89}$$

where h is Planck's constant, k is Boltzmann's constant, T is the absolute temperature, and f is the frequency in cycles per second.

The incremental number of such energy-bearing modes of vibration is given by the count of normal frequencies lying between f and $f + df$; and this is given, as in the theory of room acoustics,[1] by

$$dN = \left(\frac{4\pi Vf^2}{c^3} + \frac{\pi Sf}{2c^2} + \frac{L}{2c}\right) df \tag{3c-90}$$

where V is the volume, S the total surface, and L the sum of the three dimensions of the region under consideration, and where the three terms represent, respectively, the normal-frequency "points" distributed throughout the volume, over the coordinate planes, and along the coordinate axes of an octant of frequency space. If the three dimensions of the region are not too disparate, S can be approximated by $6V^{\frac{2}{3}}$, and L by $3V^{\frac{1}{3}}$, giving

$$dN = \frac{4\pi Vf^2 df}{c^3}\left[1 + \frac{3\lambda}{4V^{\frac{1}{3}}} + \frac{3\lambda^2}{8\pi V^{\frac{2}{3}}}\right] \tag{3c-91}$$

For sufficiently high frequencies, this reduces to the classical expression (Rayleigh, 1900; Jeans, 1905) for the distribution of normal frequencies,

$$dN = \frac{4\pi Vf^2 df}{c^3} \tag{3c-92}$$

an aymptotic form that can be shown (Weyl, 1911) to be independent of the shape of V and rigorously valid in the limit when $\lambda = c/f$ becomes small in comparison with $V^{\frac{1}{3}}$.

If attention is confined for the moment to finite frequency bands that do not include the lower frequencies, the incremental translational energy density of thermal agitation will be given by the product of (3c-89) and (3c-92). Then, by hypothesis, this can be set equal to the incremental energy density of the diffuse sound field, which is given by $d(\langle p^2\rangle/\rho c^2)$, where p is the rms sound pressure; thus

$$d\frac{\langle p^2\rangle}{\rho c^2} = \frac{(4\pi f^2 df/c^3)hf}{\exp(hf/kT) - 1} \tag{3c-93}$$

$$= \frac{(4\pi kT/c^3)f^2 df(hf/kT)}{\exp(hf/kT) - 1}$$

$$= \frac{4\pi kT}{c^3} f^2 df\left[1 - \frac{1}{2}\frac{hf}{kT} + \frac{1}{12}\left(\frac{hf}{kT}\right)^2 - \cdots\right], \quad \left(\frac{hf}{kT}\right)^2 < 4\pi^2 \tag{3c-94}$$

The total energy density associated with all the allowed modes of vibration is then to be found by extending the integral of (3c-94) over all frequencies less than the upper limiting frequency for which the mode count [by (3c-92)] is just equal to three times n_V, the total number of molecules in unit volume. This upper frequency limit, f_{lim}, is given, for either liquids or gases, by the integral of (3c-92);

$$\frac{N_{\text{lim}}}{V} = \frac{4\pi f_{\text{lim}}{}^3}{3c^3} = 3n_V = 3A\frac{\rho}{M} \qquad f_{\text{lim}}{}^3 = \frac{9c^3 A\rho}{4\pi M} \tag{3c-95}$$

where A is Avogadro's number (6.025×10^{26} molecules/kg mole), ρ is in kg/m³, and M is the molecular weight (numeric, $O_2 = 32$). At ordinary room temperature, $f_{\text{lim}} \approx 2 \times 10^{10}$ c/s for air, $\approx 4 \times 10^{12}$ c/s for water. These frequencies are well outside the range so far accessible for acoustical experimentation and need not be

[1] Maa, *J. Acoust. Soc. Am.* **10**, 235–238 (1939); Bolt, *J. Acoust. Soc. Am.* **10**, 228–234 (1939).

considered further except when the foregoing notions are used as the basis for a theory of specific heats, in which case it is necessary also to take into account vibrational and rotational degrees of freedom, and to reexamine the equilibrium statistics that underlie (3c-89). Note in passing that the *phonon* of specific-heat theory merely identifies the burden of internal energy carried by each of the normal modes of vibration postulated above.

Within the ranges of frequency and temperature ordinarily of interest in the assessment of thermal noise, the exponent hf/kT is so small that even the linear term in the series expansion of (3c-94) can be omitted. This amounts to a reversion to the classical analysis of energy partition in continuous media[1] and to the assignment of an energy kT to each allowed mode of vibration. With this simplification, (3c-94) can be integrated at once to yield the mean-square sound pressure, in the frequency band $f_2 - f_1$, as

$$\langle p^2 \rangle = \frac{4}{3}\pi kT \frac{\rho}{c}(f_2{}^3 - f_1{}^3) \qquad (\text{newtons/m}^2)^2 \qquad (3c\text{-}96)$$

in which Boltzmann's constant $k = 1.380 \times 10^{-23}$ joule/deg Kelvin, T is in degrees Kelvin, ρ in kg/m^3, and c in m/sec. To facilitate computation, it is useful to rearrange (3c-96) in the following forms:

$$p_{\text{rms}} = 1.3 \times 10^{-12}\left(\frac{\rho}{c}\right)^{\frac{1}{2}}\left[\frac{T}{293}(f_2{}^3 - f_1{}^3)\right]^{\frac{1}{2}} \qquad \text{newtons/m}^2 \qquad (3c\text{-}97a)$$

$$(p_{\text{rms}})_{\text{air}} = 0.76 \times 10^{-10}\left[\frac{T}{293}(f_2{}^3 - f_1{}^3)\right]^{\frac{1}{2}} \qquad \text{dynes/cm}^2 = \mu\text{b} \qquad (3c\text{-}97b)$$

$$(p_{\text{rms}})_{\text{sea water}} = 10.6 \times 10^{-10}\left[\frac{T}{293}(f_2{}^3 - f_1{}^3)\right]^{\frac{1}{2}} \qquad \mu\text{b} \qquad (3c\text{-}97c)$$

in which the constants have been adjusted to make the temperature factor reduce to unity at 20°C, and where ρ/c has been taken as 0.00345 for air and 0.67 for sea water. It follows, for example, that the rms thermal noise pressure, for the wide-range audio-frequency band extending to 19 kc/s in air, is just equal to the reference sound pressure, $p_0 = 0.0002\ \mu$b.

The power spectrum of thermal noise can be deduced from either (3c-94) or (3c-97*b*) and may be expressed as a *sound spectrum level* by writing

$$\begin{aligned}\beta_{\text{noise}} &= 10\log_{10}\left(\frac{d(\langle p^2\rangle/p_0{}^2)}{df}\right) = 10\log_{10}\frac{4\pi kTf^2\rho}{cp_0{}^2}\\ &= 10\log_{10}\left[4.33 \times 10^{-7}(f_{\text{kc/s}})^2\frac{T}{293}\right]\\ &= -63.6 + 20\log_{10} f_{\text{kc/s}} + 10\log_{10}\frac{T}{293} \qquad \text{db} \qquad (3c\text{-}98)\end{aligned}$$

Note that this noise spectrum is *not* "white" but has instead a uniform positive slope of 6 db/octave, corresponding to an rms thermal-noise sound pressure that is directly proportional to frequency. On the other hand, for frequencies low enough to make the additive "correction" terms of (3c-91) significant, the noise spectrum level tends increasingly to lie *above* the +6 db/octave line as the frequency approaches the low-frequency cutoff at which only the gravest mode of vibration can be excited. The noise spectrum level can also be expected to vary erratically as the low-frequency limit is approached and the population of normal frequencies becomes sparse, in much the same way that the steady-state pressure response of small rooms varies irregularly with frequency when only a few normal modes of vibration are available for excitation. It does not follow, however, that thermal noise in such a small enclosure could be

[1] Jeans, "Dynamical Theory of Gases," 2d ed., pp. 381–391.

"quieted" by the application of sound absorbents. The boundary surfaces, without regard for their acoustical character, will always reach the same radiative equilibrium with the interior medium if both are at the same temperature; otherwise there would be a net flow of thermal "noise" energy across the boundaries in the guise of ordinary heat transfer.

The possibility that thermal noise might be the factor that limits human hearing acuity can be assessed with the help of (3c-98). If the critical-band theory of masking by wide-band noise continues to hold for subliminal stimuli, the effective masking level of thermal noise can be found by adding, at any frequency, the critical bandwidth (expressed as $10 \log_{10} \Delta f_c$) and the spectrum level given by (3c-98). Comparing this result with the binaural threshhold for random incidence then leads to the conclusion that thermal noise remains about 11 to 13 db below threshhold at the frequency of greatest vulnerability (ca. 3 to 5 kcs), even for young people with exceptionally acute hearing. On this basis human hearing might be assigned a "noise figure" of approximately 12 db. It is probable that some at least of this failure to achieve ideal function can be ascribed to internal noise of physiological origin. The near miss on thermal noise limiting gives comforting reassurance, however, that not more than a few decibels of additional hearing acuity could be utilized effectively by humans even if biological adaptation were to make it available.

3d. Acoustic Properties of Gases

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A number of the physical properties of a gas are important in determining its acoustic characteristics. These include density, pressure, temperature, specific heats, coefficients of viscosity, etc. These properties, and others, are presented and discussed below in detail.

3d-1. Density. The density ρ_0 of a number of common gases at standard temperature and pressure is given in Table 3d-1. The density at any temperature and pressure can be obtained from the expression

$$\rho = \rho_0 \left(\frac{P}{760}\right)\left(\frac{273.16}{T}\right)$$

where P is the barometric pressure in millimeters of mercury and T is the absolute temperature in degrees Kelvin.

3d-2. Atmospheric Pressure and Temperature. The atmospheric pressure and air temperatures, and consequently the air density, vary with elevation above the surface of the earth. Table 3d-2 gives the air pressure, temperature, and density as a function of elevation as compiled by Humphreys[1] and others where indicated.

[1] "Handbook of Chemistry and Physics," 37th ed. Chemical Rubber Publishing Company, Cleveland, 1954–1955.

TABLE 3d-1. DENSITY ρ_0 (0°C, 1 atm)

Gas	Formula	ρ_0, g/liter	ρ_0, lb/ft^3
Air		1.2929	0.08071
		1.2920 S	0.0806 S
Ammonia	NH_3	0.7710	0.04813
		0.7598 S	0.04742 S
		0.7708 C	0.0482 C
Argon	A	1.7837	0.11135
		1.782 S	0.1112 S
		1.7828 C	0.1114 C
Carbon dioxide	CO_2	1.9769	0.12341
		1.9630 S	0.1225 S
Carbon monoxide	CO	1.2504	0.07806
		1.2492 S	0.0779 S
Chlorine	Cl_2	3.214	0.2006
		3.1638 S	0.1974 S
		3.2204 C	0.2011 C
Ethane	C_2H_6	1.3566	0.08469
Ethylene	C_2H_4	1.2604	0.07868
Helium	He	0.17847	0.01114
Hydrogen	H_2	0.08988	0.005611
Hydrogen sulfide	H_2S	1.539	0.09608
		1.5203 S	0.0949 S
Methane	CH_4	0.7168	0.04475
		0.7152 S	0.04462 S
Neon	Ne	0.90035	0.05621
		0.8713 C	0.0544 C
Nitric oxide	NO	1.3402	0.08367
		1.3388 S	0.0836 S
Nitrogen	N_2	1.25055	0.07807
		1.2568 S (atm)	0.07846 S
		1.2499 S (chem)	0.07803 S
Nitrous oxide	N_2O	1.9778	0.1235
Oxygen	O_2	1.42904	0.08921
		1.4277 S	0.08915 S
Propane	C_3H_8	2.0096	0.1254
		2.020 S	0.1261 S
Sulfur dioxide	SO_2	2.9269	0.1827
		2.858 S	0.1784 S
Steam (100°)	H_2O	0.5980	0.0373

S = Smithsonian Tables, 9th ed., 1954.
C = J. H. Perry, "Chemical Engineers' Handbook," 3d ed., McGraw-Hill Book Company, Inc., New York, 1950.

At 0°C a 760-mm column of mercury exerts a pressure of 1.01325×10^6 dynes/cm^2. This is standard atmospheric pressure. When determining the atmospheric pressure using a mercury barometer, account must be taken of the thermal expansion of mercury, and the thermal expansions of the glass container and metallic scale.

3d-3. Specific Heat. For several common gases the values of C_p, the specific heat at constant pressure, and γ, the ratio of C_p to C_v, are given in Table 3d-3. C_v is the specific heat at constant volume. C_p is expressed in calories per gram.

TABLE 3d-2. ATMOSPHERIC PRESSURE, TEMPERATURE, AND DENSITY AS A FUNCTION OF ELEVATION*†

Elevation		Summer			Winter		
Km	Miles	Temp., °C	Pressure, mm Hg	Density dry air, g/cm^3	Temp., °C	Pressure, mm Hg	Density dry air, g/cm^3
20.0	12.4	−51.0	44.1	0.000092	−57.0	39.5	0.000085
19.0	11.8	−51.0	51.5	0.000108	−57.0	46.3	0.000100
18.0	11.2	−51.0	60.0	0.000126	−57.0	54.2	0.000117
17.0	10.6	−51.0	70.0	0.000146	−57.0	63.5	0.000137
16.0	9.9	−51.0	81.7	0.000171	−57.0	74.0	0.000160
15.0	9.3	−51.0	95.3	0.000199	−57.0	87.1	0.000187
14.0	8.7	−51.0	111.1	0.000232	−57.0	102.1	0.000220
13.0	8.1	−51.0	129.6	0.000270	−57.0	119.5	0.000257
12.0	7.5	−51.0	151.2	0.000316	−57.0	140.0	0.000301
11.0	6.8	−49.5	176.2	0.000366	−57.0	164.0	0.000353
10.0	6.2	−45.5	205.1	0.000419	−54.5	192.0	0.000408
9.0	5.6	−37.8	237.8	0.000470	−49.5	224.1	0.000466
8.0	5.0	−29.7	274.3	0.000524	−43.0	260.6	0.000526
7.0	4.3	−22.1	314.9	0.000583	−35.4	301.6	0.000590
6.0	3.7	−15.1	360.2	0.000649	−28.1	347.5	0.000659
5.0	3.1	− 8.9	410.6	0.000722	−21.2	398.7	0.000735
4.0	2.5	− 3.0	466.6	0.000803	−15.0	455.9	0.000821
3.0	1.9	+ 2.4	528.9	0.000892	− 9.3	519.7	0.000915
2.5	1.6	+ 5.0	562.5	0.000942	− 6.7	554.3	0.000967
2.0	1.2	+ 7.5	598.0	0.000990	− 4.7	590.8	0.001023
1.5	0.9	+10.0	635.4	0.001043	− 3.0	629.6	0.001083
1.0	0.6	+12.0	674.8	0.001100	− 1.3	670.6	0.001146
0.5	0.3	+14.5	716.3	0.001157	0.0	714.0	0.001215
0.0	0.0	+15.7	760.0	0.001223	+ 0.7	760.0	0.001290

* "Handbook of Chemistry and Physics," 37th ed.
† See also Sec. 2m-8, pp. 2-127 to 2-128.

3d-4. Viscosity. The coefficient of viscosity η of a number of gases is given in Table 3d-4. The units of η are dyne-seconds per square centimeter or poises.

The ratio η/ρ of viscosity to density occurs frequently and is known as the kinematic viscosity coefficient. It is usually designated by the letter ν, and has the dimensions square centimeters per second, in the cgs system. For air, $\nu = 0.151$ cm²/sec at 18°C and 760 mm of mercury.

For a plane acoustic wave propagating in an unbounded gas a small attenuation will occur because of viscosity. The attenuation factor is $e^{-\alpha_\eta x}$ for the pressure (or particle velocity) and

$$\alpha_\eta = \frac{2}{3}\frac{\eta}{\rho}\frac{\omega^2}{c^3} = \frac{2}{3}\nu\frac{\omega^2}{c^3}$$

where c is the speed of sound and ω the angular frequency of the wave.

3d-5. Thermal Conductivity. The thermal conductivity κ of a number of gases is given in Table 3d-5. The units of κ are calories per centimeter-second-degree.

The quantity $\kappa/\rho C_v$ frequently appears in heat-conduction equations. It is often designated by the symbol α, and is called the coefficient of temperature exchange.

TABLE 3d-3. SPECIFIC HEAT AT CONSTANT PRESSURE C_p AND THE RATIO γ OF C_p TO THE SPECIFIC HEAT AT CONSTANT VOLUME C_v*

[C_p (cal/g deg); $\gamma = C_p/C_v$]

Gas	Temp., °C	C_p	Temp., °C	γ
Air	−120 (10 atm)	0.2719	−118 (1 atm)	1.415
	(20 atm)	0.3221		
	(40 atm)	0.4791	+ 17 (1 atm)	1.403
	(70 atm)	0.7771	− 78 (1 atm)	1.408
	− 50 (10 atm)	0.2440	− 79 (25 atm)	1.57
	(20 atm)	0.2521	− 79 (100 atm)	2.20
	(40 atm)	0.2741		
	(70 atm)	0.3121		
	0 (1 atm)	0.2398	0 (1 atm)	1.403
	(20 atm)	0.2484	0 (25 atm)	1.47
	(60 atm)	0.2652	0 (50 atm)	1.53
	50 (20 atm)	0.2480	0 (75 atm)	1.59
	(100 atm)	0.2719	17 (1 atm)	1.403
	(220 atm)	0.2961	20 (3 atm)	1.41
	100 (1 atm)	0.2404	100 (1 atm)	1.401
	(20 atm)	0.2471		
	(100 atm)	0.2600	200 (1 atm)	1.398
	(220 atm)	0.2841		
	400 (1 atm)	0.2430	400 (1 atm)	1.393
	1000 (1 atm)	0.2570	1000 (1 atm)	1.365
	1400 (1 atm)	0.2699	1400 (1 atm)	1.341
	1800 (1 atm)	0.2850	1800 (1 atm)	1.316
Ammonia	15 (1 atm)	0.5232	15 (1 atm)	1.310
Argon	15 (1 atm)	0.1253	15 (1 atm)	1.668
Carbon dioxide	15 (1 atm)	0.1989	15 (1 atm)	1.304
Carbon monoxide	15 (1 atm)	0.2478	15 (1 atm)	1.404
Chlorine	15 (1 atm)	0.1149	15 (1 atm)	1.355
Ethane	15 (1 atm)	0.3861	15 (1 atm)	1.22
Ethylene	15 (1 atm)	0.3592	15 (1 atm)	1.255
Helium	−180 (1 atm)	1.25	−180 (1 atm)	1.660
Hydrogen	15 (1 atm)	3.389	15 (1 atm)	1.410
Hydrogen sulfide	15 (1 atm)	0.2533	15 (1 atm)	1.32
Methane	15 (1 atm)	0.5284	15 (1 atm)	1.31
Neon	....	......	19 (1 atm)	1.64
Nitric oxide	15 (1 atm)	0.2329	15 (1 atm)	1.400
Nitrogen	15 (1 atm)	0.2477	15 (1 atm)	1.404
Nitrous oxide	15 (1 atm)	0.2004	15 (1 atm)	1.303
Oxygen	15 (1 atm)	0.2178	15 (1 atm)	1.401
Propane	....	......	16 (0.5 atm)	1.13
Steam	100 (1 atm)	0.4820	100 (1 atm)	1.324
Sulfur dioxide	15 (1 atm)	0.1516	15 (1 atm)	1.29

* "Handbook of Chemistry and Physics," 37th ed.

The reciprocal of α is often called diffusivity. In the cgs system the units of α are square centimeters per second. For air $\alpha = 0.27$ cm²/sec at 18°C and 760 mm of mercury.

A plane acoustic wave propagating in an unbounded gas will be attenuated slightly

TABLE 3d-4. COEFFICIENT OF VISCOSITY η FOR DIFFERENT GASES AS A FUNCTION OF TEMPERATURE*

Gas	Formula	Temp., °C	Viscosity, micropoises (dyne-sec/cm² × 10^{-6})
Air		−31.6	153.9
		0	170.8
		18	182.7
		40	190.4
		54	195.8
		74	210.2
		100	217.5
		150	238.5
		200	258.2
		300	294.6
		400	327.7
		500	358.3
Argon	A	0	209.6
		23	221.0
Carbon dioxide	CO_2	0	139.0
		20	148.0
		40	157.0
Carbon monoxide	CO	0	166
		15	172
		100	210
Helium	He	0	186.0
		20	194.1
Hydrogen	H_2	0	83.5
		20.7	87.6
Neon	Ne	20	311.1
Nitric oxide	NO	0	178
		20	187.6
Nitrogen	N_2	27.4	178.1
Nitrous oxide	N_2O	0	135
Oxygen	O_2	0	189
		19.1	201.8
		127.7	256.8

* "Handbook of Chemistry and Physics," 37th ed., and "International Critical Tables."

because of thermal-conduction effects. The attenuation constant α_T is

$$\alpha_T = \frac{\kappa(\gamma - 1)\omega^2}{\gamma \rho C_v c^3}$$

where $\kappa/\rho C_v$ is the coefficient of temperature exchange, γ the ratio of specific heats, c the propagation velocity, and ω the angular frequency of the wave.

3d-6. Speed (Velocity) of Propagation. The speed of sound for small sound amplitudes can be written exactly as[1]

$$c = \left[\frac{RT}{M}\left(f + \frac{gR}{hC_V^\infty}\right)\right]^{\frac{1}{2}}$$

where

$$f = -\frac{V^2}{RT}\left(\frac{\partial p}{\partial V}\right)_T$$

$$g = \left(\frac{V}{R}\frac{\partial p}{\partial T}\right)_V^2$$

$$h = \frac{C_v}{C_v^\infty} = 1 + \frac{T}{C_v^\infty}\int_\infty^V \left(\frac{\partial^2 p}{\partial T^2}\right)_v dV$$

C_v^∞ is the specified heat for constant volume as the volume approaches infinity; M, the molecular weight of the gas, has been substituted for ρV; and R, the gas constant, puts the equation in a useful form. The quantities f, g, h are dimensionless and differ only slightly from unity as determined by the imperfection of the gas.

TABLE 3d-5. THERMAL CONDUCTIVITY κ OF GASES AT 0°C*

Gas	Formula	Thermal conductivity κ at 0°C (cal/cm-sec-deg)
Air	....	0.0548×10^{-3}
Argon	A	0.0387×10^{-3}
Carbon dioxide	CO_2	0.0340×10^{-3}
Helium	He	0.344×10^{-3}
Hydrogen	H_2	0.416×10^{-3}
Neon	Ne	0.1104×10^{-3}
Nitrogen	N_2	0.0566×10^{-3}
Oxygen	O_2	0.0573×10^{-3}
Steam (100°C)	H_2O	0.0551×10^{-3} (100°C)

* Kennard, "Kinetic Theory of Gases," McGraw-Hill Book Company, Inc., New York, 1938.

Thus if the molecular weight, the specific heat, and the equation of state are known, the velocity of sound under any conditions can be calculated.

For an ideal gas, where $PV = RT$ one can write

$$c = \left[\frac{RT}{M} - \left(1 + \frac{R}{C_v}\right)\right]^{\frac{1}{2}} = \left(\frac{RT\gamma}{M}\right)^{\frac{1}{2}} = \left(\frac{\gamma p}{\rho}\right)^{\frac{1}{2}}$$

where $\gamma = \dfrac{C_p}{C_v}$

The accepted value of c_0, the velocity at standard conditions of temperature and pressure, for a number of gases is given in Table 3d-6.

The accepted value of the speed of sound in air, c, as calculated and checked on the average by several reported determinations is[1]

$$c_0 = 33{,}145 \pm 5 \text{ cm/sec}$$
$$c_0 = 1{,}087.42 \pm 0.16 \text{ fps}$$

under the conditions (1) audible frequency range, (2) temperature at 0°C, (3) 1 atm pressure, (4) 0.03 mole per cent content of CO_2, (5) 0 per cent water content. To

[1] See Hardy, Telfair, and Pielemeier, *J. Acoust. Soc. Am.* **13**, 226 (1942).

TABLE 3d-6. SPEED (VELOCITY) OF SOUND IN GASES*

Gas	Formula	Speed, m/sec at 0°C	Speed, fps at 0°C
Air	....	331.45	1,087.42
Ammonia	NH_3	415	1,361
Argon	A	319	1,046
Carbon monoxide	CO	337.1	1,106
Carbon dioxide	CO_2	258.0 (low freq.)	846 (low freq.)
		268.6 (high freq.)†	881 (high freq.)†
Carbon disulfide	CS_2	189	606
Chlorine	Cl_2	205.3	674
Ethylene	C_2H_4	314	1,030
Helium	He	970	3,182
Hydrogen	H_2	1,269.5	4,165
Illuminating gas	....	490.4	1,609
Methane	CH_4	432	1,417
Neon	Ne	435	1,427
Nitric oxide	NO	325	1,066
Nitrogen	N_2	337	1,096
Nitrous oxide	N_2O	261.8	859
Oxygen	O_2	317.2	1,041
Steam (100°C)	H_2O	404.8	1,328

* "Handbook of Chemistry and Physics," 37th ed., "International Critical Tables," and *J. Acoust. Soc. Am.*

† "High frequencies" means that the acoustic period is so short that the periodic changes in the vibrational heat constant cannot remain in phase with the other periodic changes as the sound wave passes through the gas.

calculate the speed of sound at various temperatures one can write

$$c = \left(\frac{R\gamma}{M} 273.16\right)^{\frac{1}{2}} \sqrt{\frac{T}{273.16}}$$

$$= 33{,}145 \sqrt{\frac{T}{273.16}} \quad \text{cm/sec}$$

$$= 33{,}145 \left(1 + \frac{°\text{C}}{273.16}\right) \quad \text{cm/sec} \quad \left(\frac{°\text{C}}{273.16} \ll 1\right)$$

where T = absolute temperature
°C = temperature, °C

If the gas is made up of a mixture of gases or if water vapor is present the expression

$$c = \left[\frac{RT}{M}\left(1 + \frac{R}{C_v}\right)\right]^{\frac{1}{2}}$$

can still be used to calculate the velocity. The molecular weight M of the mixture can be calculated, or, realizing that $RT/M = p/\rho$, the density of the mixture can be used.

In addition to correcting M (or ρ) it is necessary to correct C_v also. It is incorrect to take the weighted average of the ratio of the specific heats, γ. The weighted average of the specific heats themselves must be used.

For rough calculations of the variation with humidity or composition, it is probably sufficient merely to correct for the density of the mixture.

3d-7. Characteristic Impedance. The characteristic impedance is equal to the ratio of the sound pressure to the particle velocity in a plane wave traveling in an unbounded medium. It is equal to the density times the velocity of propagation, that is, ρc. The variation of ρc with temperature can be calculated from the expression

$$\rho c = \rho_0 c_0 \left(\frac{273.16}{T}\right)^{\frac{1}{2}} \frac{P}{760} \qquad \text{rayls}$$

where $\rho_0 c_0$ is the value at 0°C and 1 atm pressure. For air $\rho_0 c_0 = 42.86$ dyne sec/cm³. Table 3d-7 contains values of $\rho_0 c_0$ for several common gases.

TABLE 3d-7. CHARACTERISTIC IMPEDANCE $\rho_0 c$ OF COMMON GASES AT 0°C (273.16°K) TEMPERATURE AND 760 MM HG BAROMETRIC PRESSURE

Gas	Formula	$\rho_0 c_0$, dyne-sec/cm³ at 0°C, 760 mm Hg
Air	....	42.86
Argon	A	56.9
Carbon dioxide	CO_2	50.8
Carbon monoxide	CO	42.1
Helium	He	17.31
Hydrogen	H_2	11.41
Neon	Ne	38.5
Nitric oxide	NO	43.5
Nitrogen	N_2	42.1
Nitrous oxide	N_2O	51.8
Oxygen	O_2	45.3

3d-8. Attenuation. In addition to the dispersion of sound due to wind, turbulence in the atmosphere, and temperature gradients, two properties of the medium combine to attenuate a wave which is propagated in free space. The first of these attenuations is caused by molecular absorption and dispersion in polyatomic gases involving an exchange of translational and vibrational energy between colliding molecules. The second is due to viscosity and heat conduction in the medium.

Knudsen[1] says that "the attenuation of sound is greatly dependent upon location and weather conditions, that is, upon the humidity and temperature of the air. The cold air of the arctic is acoustically transparent; the attenuation of sound is not much more than that attributable to viscosity and heat conductivity; . . . for the hot and relatively dry summer air of the desert, such as at Greenland Ranch, Inyo County, California, where the relative humidity may drop as low as 2.4 per cent, the attenuation at 3000 cps is 0.14 db/m, and at 10,000 cycles it is 0.48 db/m."

Data on the absorption of audible sound in air are valuable because they are needed to calculate the reverberation time for high-frequency sound in rooms, for determining the amplification characteristics of public-address systems for use outdoors, and for predicting the range of effectiveness of apparatus for sound signaling and sound ranging in the atmosphere.

Kneser[1] has treated analytically the problem of absorption and dispersion of sound by molecular collision. He summarized his results in the form of a nomogram which has been reprinted along with comments by Pielemeier.[3] Pielemeier observes that for

[1] V. O. Knudsen, The Propagation of Sound in the Atmosphere—Attenuation and Fluctuations, *J. Acoust. Soc. Am.* **18,** 90–96 (1946).

[2] H. O. Kneser, The Interpretation of the Anomalous Sound-absorption in Air and Oxygen in Terms of Molecular Collisions, *J. Acoust. Soc. Am.* **5,** 122–126 (1933); A Nomogram for Determination of the Sound Absorption Coefficient in Air, *Akust. Z.* **5,** 256–257 (1940) (in German).

[3] W. H. Pielemeier, Kneser's Sound Absorption Nomogram and Other Charts, *J. Acoust. Soc. Am.* **16,** 273–274 (1945).

molecular absorption Kneser's theoretical values are lower than Knudsen's[1] experimental values for reasons not fully understood.

Kneser's nomogram is reproduced in Fig. 3d-1. By means of it, the attenuation due to the molecular absorption can readily be found for any ordinary set of conditions of temperature, humidity, and frequency. For example, if the temperature is 15°C, and relative humidity is 50 per cent, first locate 15° on the temperature axis, trace *left* to the 50 per cent mark, then upward to the middle of the shaded area (upper left), then to the right to the proper frequency curve (3 kc in this case), then downward to the K scale. Next begin another tracing at 15°C toward the *right* until the lower right curve is reached, then trace upward to the log $(M) + 7$ scale. Then join the

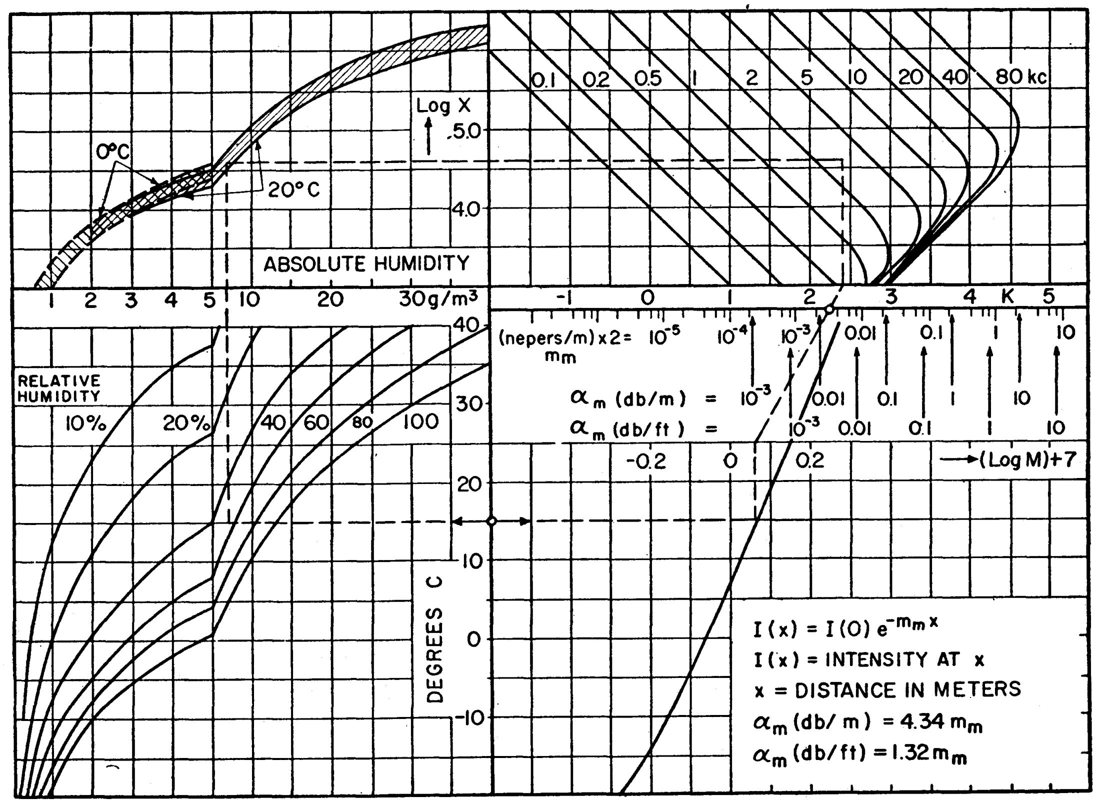

FIG. 3d-1. Nomogram for determining the attenuation in air caused by molecular absorption. (*From L. L. Beranek, "Acoustics Measurement," John Wiley & Sons, Inc., New York, 1949; after Kneser.*)

end points of the two tracings with a straight line. The value of the molecular attenuation α_m as read on that scale will be 12 db/km or 3.7×10^{-3} db/ft. The half width of the shaded band in the log X chart of Fig. 3d-1 represents the uncertainty in the log X values. Note that the band changes position slightly with temperature.

The attenuation caused by heat conduction and viscosity of the air α_c is not known so accurately. The classical absorption due to these causes has been thoroughly described by Lord Rayleigh[2] and was first derived by Kirchhoff and Stokes as the relation

$$\alpha_c = \alpha_\eta + \alpha_T = \frac{\omega^2}{2\rho_0 c^3}\left[\frac{4\eta}{3} + (\gamma - 1)\frac{\kappa}{C_p}\right] \qquad \text{nepers/cm}$$

where $\omega/2\pi$ = frequency in cycles per second; ρ_0 = density in grams per centimeter

[1] V. O. Knudsen, The Absorption of Sound in Air, in Oxygen and in Nitrogen—Effects of Humidity and Temperature, *J. Acoust. Soc. Am.* **5**, 112–121 (1933).

[2] Lord Rayleigh, "Theory of Sound," The Macmillan Company, New York, 1929.

cubed; c = speed of sound in centimeters per second; η = coefficient of viscosity in poises; γ = ratio of specific heats; κ = coefficient of thermal conductivity in calories per second-degree-centimeter; and C_p is the specific heat at constant pressure in calories per gram-degree.

Recent papers by Sivian[1] and Krasnooshkin[2] have led to somewhat higher values for the absorption caused by viscosity. The data from these three sources are given by Fig. 3d-2 and the equations

For λ in feet, $$\alpha_c = 0.143 \frac{A}{\lambda^2} \quad \text{db/ft}$$

For λ in meters, $$\alpha_c = 0.0437 \frac{A}{\lambda^2} \quad \text{db/m}$$

where λ is the wavelength and A is given in the curve in Fig. 3d-2.

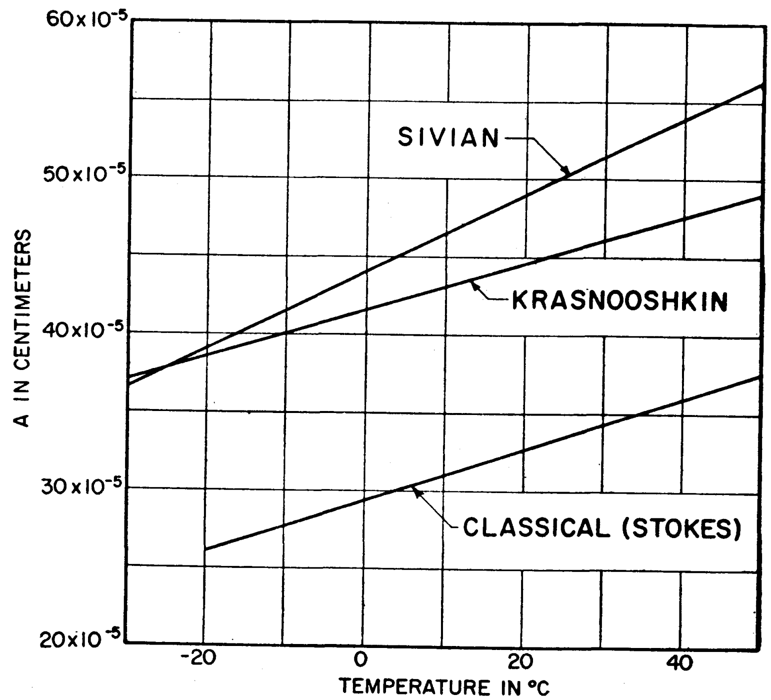

FIG. 3d-2. Plot of A in centimeters as a function of temperature: $A = \alpha_c\lambda^2/0.0437$, where λ = wavelength in meters, and α_c is the attenuation constant in db per meter for a free-traveling plane wave. The upper line (Sivian) obtained by multiplying the Stokes value by 1.5, lies closer to measured values than does either of the other two. *(From L. L. Beranek, "Acoustic Measurements," John Wiley & Sons, Inc., New York, 1949.)*

The total attenuation α_A due to both types of absorption is therefore

$$\alpha_A = \alpha_m + \alpha_c \quad \text{db/ft (or db/m)}$$

These high values of attenuation appear to come from the H_2O vapor content of the air, although they cannot be calculated accurately by the Kneser nomogram. At frequencies above 100 kc for undried air and at all frequencies for dried air, and oxygen and nitrogen, the measured attenuation is about 1.5 times that predicted by the Stokes relation.

[1] L. J. Sivian, High Frequency Absorption in Air and in Other Gases," *J. Acoust. Soc. Am.* **19,** 914–916 (1947).

[2] P. E. Krasnooshkin, On Supersonic Waves in Cylindrical Tubes and the Theory of the Acoustical Interferometer, *Phys. Rev.* **65,** 190 (1944). See also W. H. Pielemeier, Observed Classical Sound Absorption in Air, *J. Acoust. Soc. Am.* **17,** 24–28 (1945).

Some experimental values by Knudsen and Harris[1] for the total attenuation α_A at room temperature and for various values of relative humidity are given in Fig. 3d-3.

An empirical equation, which describes the measured values of Knudsen and Harris with good accuracy for relative humidities above 30 per cent and at a temperature of 20°C, is given by Cremer[2]

$$\alpha_A = \left(\frac{f}{1{,}000}\right)^{\frac{3}{2}} \frac{0.28}{20 + \phi_{20}} \qquad \text{db/m}$$

where ϕ_{20} is the relative humidity at 20°C and f is the frequency. For temperatures

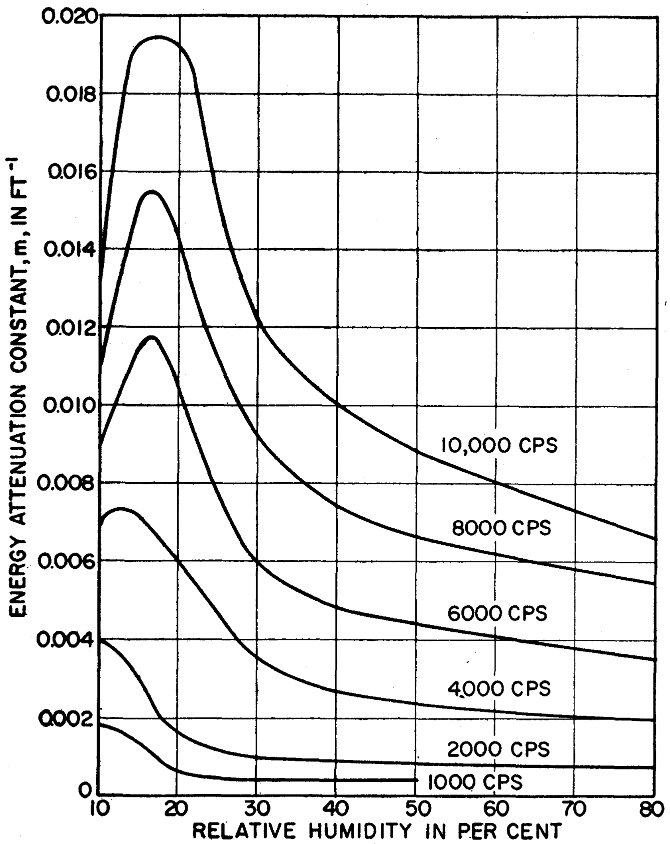

FIG. 3d-3. Measured values of the energy attenuation constant m as a function of relative humidity for different frequencies, $I(x) = I_0 \exp(-mx)$. The temperature is assumed to be about 68°F. *(From L. L. Beranek, "Acoustics," McGraw-Hill Book Company, Inc., New York, 1954: after* V. O. *Knudsen and C. M. Harris, Acoustical Designing in Architecture, p. 160, Fig. 8.10, John Wiley & Sons, Inc., New York, 1950.)*

differing slightly from 20°C, the measured value of relative humidity should be corrected to give a value of ϕ_{20} to be used in the above equation;

$$\phi_{20} = \phi_t(1 + 0.067\Delta t)$$

where Δt denotes temperature departure from 20°C. The quantity α_A is 4.34m, in the same units of distance.

[1] V. O. Knudsen and C. M. Harris, "Acoustical Designing in Architecture," p. 160 Fig. 8.10, John Wiley & Sons, Inc., New York, 1950.

[2] Lothar Cremer, "Die wissenschaftlichen Grundlagen der Raumakustik" (The Scientific Foundations of Room Acoustics), vol. III, S. Hirzel Verlag, Leipzig, 1950.

3e. Acoustic Properties of Liquids

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3e-1. Symbols and Definitions. Unless otherwise specified, cgs units are used.

- ρ density
- k adiabatic compressibility
- f frequency
- ω $2\pi f$
- c speed of propagation of plane (or spherical) waves (velocity of sound), $c = (1/\rho k)^{\frac{1}{2}}$
- ρc characteristic impedance
- α coefficient of absorption, nepers/cm
  For viscous absorption $\alpha \simeq \omega^2(2n + n')/2\rho c^3$ where n = shear viscosity and n' = dilatational viscosity.
- α/f^2 absorption constant
- db/dd intensity loss in decibels if distance from source is doubled. This unit is usually used only when loss is due to the geometry of the sound field as in spherical or cylindrical waves.

3e-2. Acoustical Comparison between Liquids and Gases. The acoustical behavior of liquids is fundamentally identical to that of gases, but the great differences in the magnitudes of the basic properties, density and compressibility, give rise to notable differences in the nature of practical sound fields in the two media. Thus the techniques which have been developed for the study of sound in gases cannot generally be applied successfully to the study of sound in liquids.

Numerically, the characteristic impedance ρc of liquids is three to four orders of magnitude greater than that of gases. Thus a liquid-gas interface appears as a substantially rigid boundary to a sound in the gas but as an effective pressure-release surface to a sound in the liquid. Even a thin film of gas, or a multiplicity of gaseous bubbles, generally prevents the existence of appreciable sound pressure in the neighboring liquid.

The compliance of solid boundaries is usually negligible compared with the compressibility of gases but is usually appreciable compared with the compressibility of liquids. Thus the simple types of sound field which are readily obtained in a gaseous medium by virtue of effectively rigid boundaries are extremely difficult to realize in a liquid medium. Types of sound fields from which acoustical properties of liquids can be determined have usually been obtained in the laboratory at high frequencies. Most of the published data on such properties were obtained in the megacycle frequency region.

3e-3. Sound Transmission in Large Bodies of Water. Sound transmission at sea is influenced largely by three factors: the geometry of the sound field, the nature of the upper and lower boundaries, and refraction. At short ranges, if source and receiver are at sufficient depths, spherical spreading of sonic energy is approximated and the intensity varies nearly as the inverse square of the range (6 db/dd). At long ranges, the field roughly approximates a two-dimensional continuum producing cylindrical spreading whereby the intensity tends to vary as the inverse first power of the range (3 db/dd). There is an intermediate range in which the controlling factor may be the interference between the direct sound and the sound reflected from the surface. For sinusoidal sound this interference produces the Lloyd-mirror effect. For broad bands the intensity tends to vary as the inverse fourth power of the range (12 db/dd).

These trends have been observed. They are dependent on such factors as source depth, receiver depth, depth of water, smoothness of surface, smoothness and reflectivity of bottom, frequency spectrum, and directivities of transducers. The trends are modified—sometimes completely masked—by the effects of refraction.

Refraction is caused by gradients in temperature, salinity, density, and currents. A major effect is a nonuniform distribution of sonic energy, frequently resulting in shadow zones and skip distances. At times sound channels are formed, i.e., layers within which the sound is trapped by virtue, for example, of downward refraction near the surface due to a temperature gradient and upward refraction in deeper water due to the density gradient.

Reverberation in water is produced by the scattering of sound by minute particles of suspended matter, marine life, and other inhomogeneities. Reverberation due to that portion of the sound which is scattered by the top and bottom surfaces is sometimes called "surface reverberation."

3e-4. Cavitation. The American Standard Acoustical Terminology gives the definition (Z24.1, 9.035): "Cavitation is the formation of local cavities in a liquid as a result of the reduction of total pressure." Cavitation may occur as the result of a sound-wave rarefaction, such as is produced in the negative pressure cycle of an underwater transmitting transducer, or as the result of the reduction of pressure due to hydrodynamic flow, such as is produced by the movement at high speed of a propeller underwater. Broad-band noise is generated by cavitation; a large amount of evidence indicates that this noise is associated with the collapse of cavitation bubbles. In many instances the noise of cavitation has been observed to begin before the cavitation bubbles have been visible to the unaided eye.

In shallow water depths, since atmospheric pressure corresponds to but a low hydraulic head in liquids, cavitation may occur at moderate sound intensities. Numerically, at a static pressure of N atm, the intensity of a sinusoidal plane (or spherical) wave in water at which the total pressure becomes zero at a negative peak is $I \simeq N^2/3$ watts/cm².

The observed cavitation threshold corresponds in many cases to a substantial negative pressure, usually reported to have a very variable value. Many degassed liquids show a tensile strength of the order of an atmosphere. Over very short time intervals this figure is much higher. The threshold of acoustically produced cavitation thus depends on the frequency. It also depends on gas content, ion content, and suspended matter (all cavity-producing nuclei), temperature, viscosity, cleanliness of the container, and the past history of the liquid.

Since cavitation bubbles reduce the sound that is radiated by a transducer, transformer oils and castor oil, which do not cavitate readily, are sometimes used to transmit sound from the transducer face to an outer radiating surface at which the intensity has been spreduced by reading.

3e-5. Dispersion. There is no firm evidence that the speed of propagation of sound in a simple liquid is dependent on frequency.

3e-6. Water and Aqueous Solutions. Table 3e-1, taken from the American Standards Association Acoustical Terminology (Z24.1-1951, Table 9.1), gives various properties of fresh and sea water under representative water conditions.

TABLE 3e-1. PROPERTIES OF FRESH AND SEA WATER

	Fresh water		Sea water			
Salinity (parts per 1,000)	0		30		36	
Temp., °C	4	25	5	20	15	25
Velocity, m/sec	1,418.3	1,493.2	1,461.0	1,513.2	1,505.0	1,532.8
Density g/cm³	1.00000	0.99707	1.02375	1.02099	1.02677	1.02412
Characteristic impedance $\times 10^{-5}$ (cgs units)	1.4183	1.4888	1.4957	1.5450	1.5453	1.5698

Hydrostatic pressure increases the velocity by 0.018 m/second per meter of depth. It also increases the density by approximately 0.0000045 g/cm³ per meter of depth.

The velocities listed in Table 3e-1 are from Kuwaharara's tables.[1] More recent measurements indicate that the velocity in sea water is 3 to 4 m/sec higher.[2]

Up to 1,000 Mc, no measurable effect of frequency on velocity has been found.

The attenuation in the pressure amplitude of a plane progressive wave is expressed by $p(x) = p_0e^{-\alpha x}$. The theoretical value of α (Stokes-Kirchhoff) for viscous absorp-

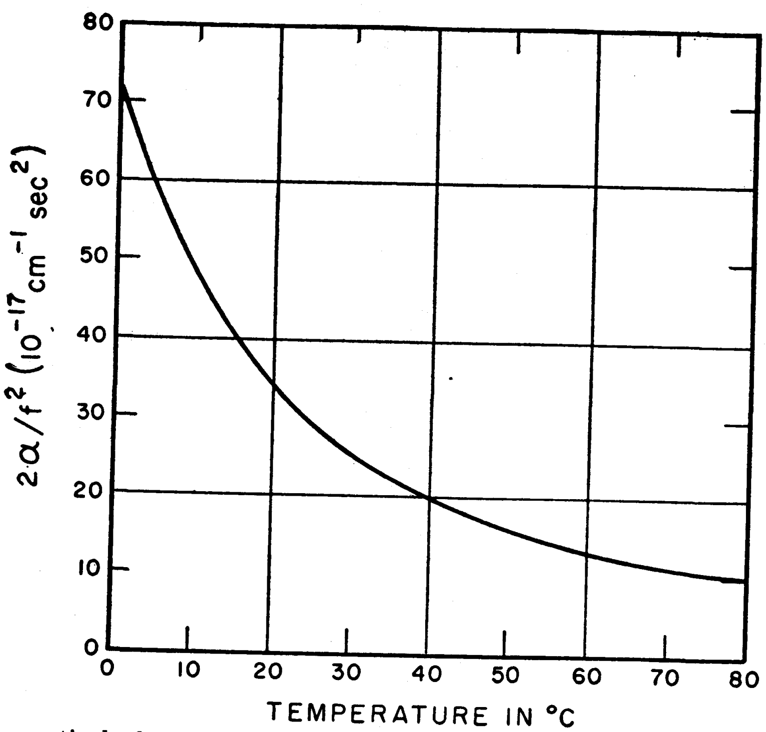


FIG. 3e-1. Theoretical absorption in water as a function of temperature. (*After Hall.*)

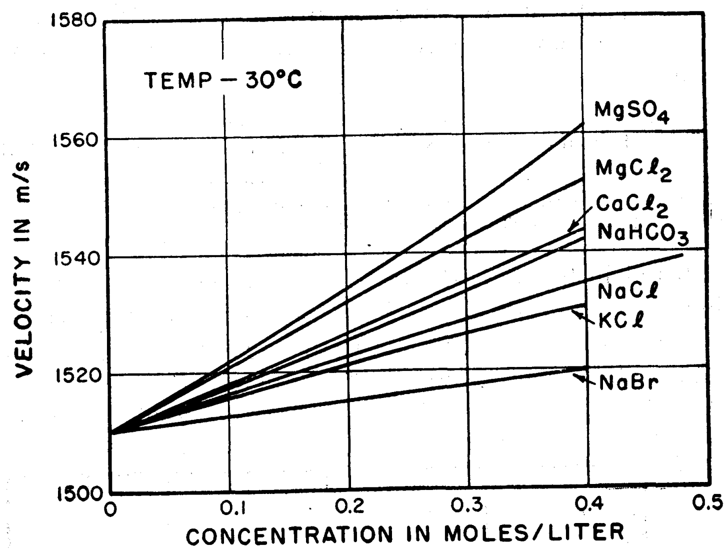
tion depends on f^2. The measured value of $\alpha/f^2 = 21.5 \times 10^{-17}$ cm^{-1} reported by Fox and Rock[3] for water has generally[4] been found to hold within experimental limits at room temperature over a very wide range of frequencies. This number has been

[1] Kuwahara, *Hydrographic Rev.* **16,** 123 (1939).
[2] Weissler and Del Grosso, *J. Acoust. Soc. Am.* **23,** 219 (1951).
[3] Fox and Rock, *J. Acoust. Soc. Am.* **12,** 505 (1941).
[4] Measured values have, however, been reported over 1,000 times greater than these.

shown by Hall[1] to correspond to the Stokes-Kirchhoff expression if bulk (dilatational) viscosity as well as shear viscosity is taken into account.

Hall's analysis includes the theoretical effect of temperature on attenuation. The values plotted in Fig. 3e-1 have been verified by several experiments.

Absorption in organic liquids shows no observable relation to the viscosity. The increments in sound velocity due to dissolved salts at the low concentrations found in sea water are found to be proportional to the molar concentration for each salt and to be additive (see Fig. 3e-2) for a number of salts.



SUBOW'S SEA WATER

SALT	CONC. (MOLAR)	Δc (VELOCITY INCREMENT)
NaCℓ	0.4649	28.2
$MgSO_4$	0.0281	3.4
$MgCℓ_2$	0.0263	2.9
$CaCℓ_2$	0.0105	0.9
KCℓ	0.0100	0.6
$NaHCO_3$	0.0025	0.2
NaBr	0.0008	0.0
		Σ = 36.2

SUBOW'S WATER OBSERVED —— 1545.8 m/s

SAME, CALCULATED BY SUMMATION — 1546.2 m/s

Fig. 3e-2. Effect of dissolved salts on sound velocity. (*After Weissler and Del Grosso.*)

The effect on absorption of dissolved solids frequently exhibits relaxation phenomena. The absorption in sea water at frequencies above 1 Mc is substantially that in fresh water. Below 70 kc the observed value of α is about 10 times greater in sea water. In the transition region from 70 to 1,000 kc, α is not proportional to f^2 (see Fig. 3e-3). This additional attenuation has been variously attributed to the high concentration (and hence partial dissociation) of NaCl and to the presence of $MgSO_4$.[2]

Figure 3e-3 indicates the observed values of absorption in sea water in the transition range.

Sound velocity and absorption in liquid mixtures exhibit two distinct types of behavior. Mixtures of organic liquids tend to have values for c and for α which vary unidirectionally (not necessarily uniformly) with the relative proportions of the

[1] Hall *Phys. Rev.* **73,** 775 (1948).
[2] Liebermann, *J. Acoust. Soc. Am.* **20,** 868 1948).

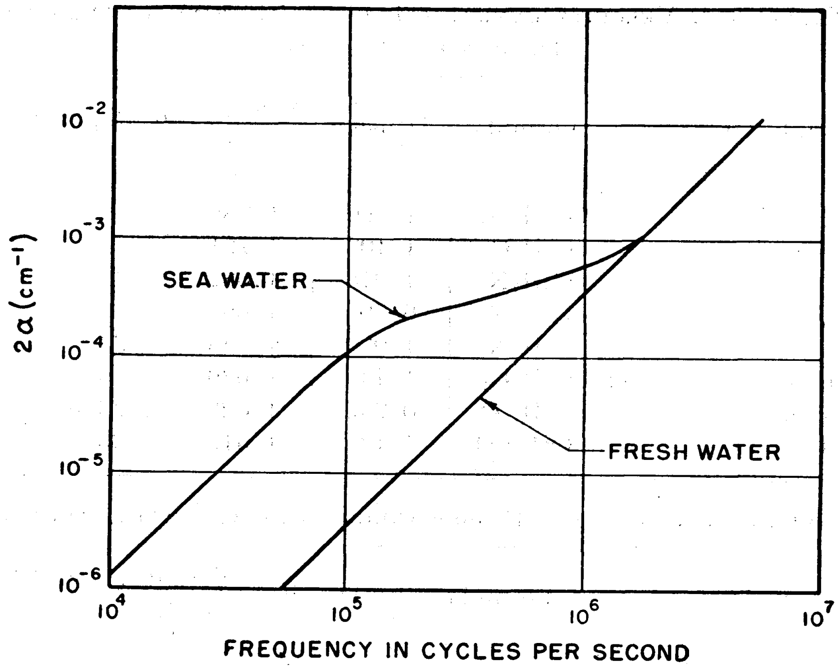


FIG. 3e-3. Sound absorption coefficients for sea water and fresh water. (*After Liebermann.*) To convert to decibels per kiloyard, multiply 2α by 3.97×10^5.

TABLE 3e-2. VELOCITY UNDER 1,100 M/SEC. LISTING IN ORDER OF INCREASING VELOCITY

Material	Formula	Density	Velocity, m/sec	Temp., °C	$\rho c \times 10^{-5}$ cgs
Ethyl bromide	C_2H_5Br	1.428	892	28	1.27
Carbon tetrachloride	CCl_4	1.596	928.5	23	1.48
Bromoform	$CHBr_3$	2.889	929	23.5	2.68
Butyl iodide (*n*)	C_4H_9I	1.616	959	28	1.55
Methylene bromide	CH_2Br_2	2.453	971	24	2.38
Methylene iodide	CH_2I_2	3.323	977	24	3.25
Butyl chloride	C_4H_9Cl	0.84	985	25	0.83
Chloroform	$CHCl_3$	1.487	1,001	23.5	1.49
Acetyl tetrabromide	$C_2H_2Br_4$	2.962	1,007	28	2.98
Ethylene bromide	$C_2H_4Br_2$	2.178	1,014	24	2.21
Butyl bromide (*n*)	C_4H_9Br	1.272	1,016	28	1.29
Acetylene dichloride	$C_2H_2Cl_2$	1.262	1,025	25	1.29
Pentane	C_5H_{12}	0.632	1,052	18	0.66
Allyl chloride	C_3H_5Cl	0.937	1,088	28	1.02

liquids. Solutions of organic liquids in water tend to show peaks in both c and α at some concentration. The velocity peaks are typically 5 to 10 per cent higher than that in either pure liquid, but the attenuation peak may show an increase of an order of magnitude over that of the organic liquid.[1]

Gases in actual solution in water are generally reported to have negligible effect on sound velocity and absorption.

[1] Willard, *J. Acoust. Soc. Am.* **12,** 438 (1941); Willis, *J. Acoust. Soc. Am.* **19,** 242 (1947); Burton, *J. Acoust. Soc. Am.* **20,** 186 (1948).

TABLE 3e-3. VELOCITY OVER 1,600 M/SEC. LISTING IN ORDER OF DECREASING VELOCITY

Material	Formula	Density	Velocity, m/sec	Temp., °C	$\rho c \times 10^{-5}$ cgs
Glycerin	$C_3H_8O_3$	1.260	1,986	22	2.50
Ethylene glycol	$C_2H_6O_2$	1.103	1,721	24	1.90
Aniline	C_6H_7N	1.018	1,682	24	1.71
Toluidine	C_7H_9N	0.994	1,669	22.5	1.66
Quinoline	C_9H_7N	1.090	1,643	22	1.79
Resorcin monomethyl ether	$C_7H_8O_2$	1.145	1,629	26	1.86
Cyclohexanol	$C_6H_{12}O$	0.946	1,622	23.5	1.53
Formamide	CH_3NO	1.13	1,610	25	1.82

TABLE 3e-4. SATURATED HYDROCARBONS AND ALCOHOLS; ACETATES

Material	Formula	Density	Velocity, m/sec	$\rho c \times 10^{-5}$ cgs
A. Saturated Hydrocarbons				
Pentane	C_5H_{12}	0.622	1,052	0.65
Hexane	C_6H_{14}	0.658	1,113	0.73
Heptane	C_7H_{16}	0.681	1,165	0.79
Octane	C_8H_{18}	0.702	1,238	0.87
B. Saturated Alcohols				
Methyl	CH_3OH	0.792	1,130	0.89
Ethyl	C_2H_5OH	0.786	1,207	0.95
Propyl	C_3H_7OH	0.801	1,234	0.99
Butyl	C_4H_9OH	0.808	1,315	1.06
Amyl	$C_5H_{11}OH$	0.813	1,347	1.09
C. Acetates				
Methyl	CH_3COOCH_3	0.928	1,211	1.12
Ethyl	$CH_3COOC_2H_5$	0.898	1,187	1.07
Propyl	$CH_3COOC_3H_7$	0.891	1,182	1.05
Butyl	$CH_3COOC_4H_9$	0.871	1,179	1.03
Amyl	$CH_3COOC_5H_{11}$	0.875	1,168	1.02

Gas bubbles in water are known to have a marked effect on both velocity and absorption.[1] The effect of air mixed in the surface water at sea by virtue of "white caps" has been found to persist after 48 hr of calm. Underwater sound measurements in the laboratory may be affected for many days by the air released from solution in tap water if not degassed.

3e-7. Acoustical Properties of Organic Liquids. The sound velocity in pure organic liquids covers little more than a 2:1 range; the lowest reported is for ethyl bromide (892 m/sec) and the highest is for glycerin (1,986 m/sec). With few excep-

[1] A. B. Wood, "A Textbook of Sound," The Macmillan Company, New York, 1941; D. T. Laird and P. M. Kendis, *J. Acoust. Soc. Am.* **24**, 29 (1952).

TABLE 3e-5. ABSOLUTE VALUES OF THE ABSORPTION CONSTANT FOR A NUMBER OF ORGANIC LIQUIDS. LISTING IN ORDER OF DECREASING ABSORPTION (Temperature between 23 and 27°C)

Material	Formula	Absorption $\alpha/f^2 \times 10^{15}$	Density	Velocity, m/sec	$\rho c \times 10^{-5}$ cgs
Carbon disulfide	CS_2	74	1.26	1,149	1.45
Glycerol	$C_3H_8O_3$	26	1.26	1,986	2.50
2, 3-Butanediol	$C_4H_{10}O_2$	20	1.05		
Benzene	C_6H_6	8.3(9.15)	0.87	1,295(1,310)	1.13
Carbon tetrachloride	CCl_4	5.7	1.59	930(928)	1.48
Cyclohexanol	$C_6H_{12}O$	5.0	0.96	1,622	1.56
Acetylene dichloride	$C_2H_2Cl_2$	4.0	1.26	1,025	1.29
Chloroform	$CHCl_3$	3.8(4.74)	1.49	995(1,001)	1.48
3-Methyl cyclohexanol resid	$C_7H_{14}O$	3.5	0.92	1,400	1.29
t-Amyl alcohol	$C_5H_{12}O$	3.3	0.81	1,204	0.975
Mesityl oxide	$C_6H_{10}O$	3.3	0.85	1,310	1.11
Bromoform	$CHBr_3$	2.3	2.89	908(929)	2.62
t-Butyl chloride	C_4H_9Cl	1.9	0.84	985	0.83
Chlorobenzene	C_6H_5Cl	1.7	1.10	1,302	1.43
Turpentine		1.5	0.88	1,255	1.10
Isopentane	C_5H_{12}	1.5	0.62	985	0.61
d-Fenchone	$C_{10}H_{16}O$	1.4	0.94	1,320	1.24
Ethyl ether	$C_4H_{10}O$	1.4(0.55)	0.71	985	0.70
Dioxane	$C_4H_8O_2$	1.3	1.03	1,380	1.42
Alkazene 13	$C_{15}H_{24}$	1.3	0.86	1,310	1.13
Kerosene		1.1	0.81	1,315	1.06
Methyl acetate	$C_3H_6O_2$	1.09	0.93	1,211	1.13
Ethyl acetate	$C_4H_8O_2$	1.1(0.77)	0.90	1,145(1,187)	1.03
Naphtha		1.0	0.76	1,225	0.93
Toluol	C_7H_8	0.9(0.85)	0.86	1,300(1,320)	1.12
Nitrobenzene	$C_6H_5NO_2$	0.9	1.20	1,490	1.79
1, 3-Dichloro-isobutane	$C_4H_8Cl_2$	0.9	1.14	1,230	1.40
Nitromethane	CH_3NO_2	0.9	1.13	1,335	1.51
Ethyl alcohol	C_2H_6O	0.9	0.79	1,150	0.91
Methyl alcohol	CH_4O	0.9	0.79	1,105(1,130)	0.87
Acetonitrile	CH_3CN	0.8	0.78	1,280(1,275)	1.00
m-Xylol	C_8H_{10}	0.78(0.74)	0.86	1,325(1,328)	1.14
Acetone	C_3H_6O	0.64(0.32)	0.79	1,170(1,203)	0.925
Alkazene 25	$C_{10}H_{12}Cl_2$	0.6	1.20	1,300	1.56
Formamide	CH_3NO	0.57	1.13	1,610	1.82
2, 5-Hexanedione	$C_6H_{10}O_2$	0.50	0.96	1,400	1.34
Water (distilled)	H_2O	0.33(0.25)	1.00	1,500(1,494)	1.50
Mercury	Hg*	0.66		1,450	

* Ring, Fitzgerald, and Hurdle, *Phys. Rev.*, **72**, 87 (1947).

tions, the range is from 1,000 to 1,500 m/sec. (It is a matter of interest that mercury also falls in this range, 1,450 m/sec.)

In contrast, the absorption constant α/f^2 varies over a wide range, about 300:1. Numerically the highest reported absorption constant for a simple liquid is about one order of magnitude lower than that for dry air.

The characteristic impedances of organic liquids ρc are distributed over the range from about 60,000 to 180,000 cgs units. Carbon tetrachloride with the ρc value of 148,000 and sound velocity of 930 m/sec is well suited for acoustic lenses in water in that the characteristic impedances are nearly matched and the velocity ratio is reasonably high, about 62:100.

The values of the properties of organic liquids reported from different sources are seldom in agreement within experimental errors. The discrepancies are presumably due to slight impurities; in the few cases in which mixtures have been investigated large effects from small concentrations have been observed.

The liquids which have been selected for tabulation are:

1. Liquids having sound velocities outside the range 1,100 to 1,600 m/sec.
2. Liquids in certain chemical groups
3. Liquids for which absorption data have been reported

The data for Tables 3e-2, 3e-3, and 3e-4 were taken from Bergman's "Ultrasonics," and for Table 3e-3 from an article by Willard.[1] It will be noted that all the organic liquids (except pentane) which have a sound velocity less than 1,100 m/sec are halogen compounds. Table 3e-2 shows that there are consistent trends within each group but inconsistent trends between groups.

In Table 3e-5 the absolute values of the absorption constant may be in error by a factor of 1.5. The relative values for liquids having nearly like properties (α/f^2 and c) should be correct within 10 per cent.

3f. Acoustic Properties of Solids

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3f-1. Elastic Constants, Densities, Velocities, and Impedances. Solids are used for conducting acoustic waves in such devices as delay lines useful for storing information, and as resonating devices for controlling and selecting frequencies. Acoustic-wave propagation in solids has been used to determine the elastic constants of single crystals and polycrystalline materials. Changes in velocity with frequency and changes in attenuation with frequency have been used to analyze various intergrain, interdomain, and imperfection motions as discussed in Sec. 3f-2.

In an infinite isotropic solid and also in a finite solid for which the wave front is a large number of wavelengths, plane and nearly plane longitudinal and shear waves can

[1] G. W. Willard, *J. Acoust. Soc. Am.* **12**, 438 (1941).

exist which have the velocities

$$v_{\text{long}} = \sqrt{\frac{\lambda + 2\mu}{\rho}} \qquad v_{\text{shear}} = \sqrt{\frac{\mu}{\rho}} \tag{3f-1}$$

where μ and λ are the two Lamé elastic moduli, μ is the shearing modulus, and $\lambda + 2\mu$ has been called the plate modulus. For a rod whose diameter is a small fraction of a wavelength, extensional and torsional waves can be propagated with velocities

$$v_{\text{ext}} = \sqrt{\frac{Y_0}{\rho}} \qquad v_{\text{tor}} = \sqrt{\frac{\mu}{\rho}}$$

where

$$Y_0 = \mu\left(\frac{3\lambda + 2\mu}{\lambda + \mu}\right) \tag{3f-2}$$

For anisotropic media, three waves will, in general, be propagated, but it is only in special cases that the particle motions will be normal and perpendicular to the direction of propagation. The three velocities satisfy an equation[1]

$$\begin{vmatrix} \lambda_{11} - \rho v^2 & \lambda_{12} & \lambda_{13} \\ \lambda_{12} & \lambda_{22} - \rho v^2 & \lambda_{23} \\ \lambda_{13} & \lambda_{23} & \lambda_{33} - \rho v^2 \end{vmatrix} = 0 \tag{3f-3}$$

where ρ is the density, v the velocity, and the λ's are related to the elastic constants of the crystal by the formulas

$$\begin{aligned}
\lambda_{11} &= l^2c_{11} + m^2c_{66} + n^2c_{55} + 2mnc_{56} + 2nlc_{15} + 2lmc_{16} \\
\lambda_{12} &= l^2c_{16} + m^2c_{26} + n^2c_{45} + mn(c_{46} + c_{25}) + nl(c_{14} + c_{56}) + lm(c_{12} + c_{66}) \\
\lambda_{13} &= l^2c_{15} + m^2c_{46} + n^2c_{33} + mn(c_{45} + c_{36}) + nl(c_{13} + c_{55}) + lm(c_{14} + c_{56}) \\
\lambda_{23} &= l^2c_{56} + m^2c_{24} + n^2c_{34} + mn(c_{44} + c_{23}) + nl(c_{36} + c_{45})lm(c_{25} + c_{46}) \\
\lambda_{22} &= l^2c_{66} + m^2c_{22} + n^2c_{44} + 2mnc_{24} + 2nlc_{46} + 2lmc_{26} \\
\lambda_{33} &= l^2c_{55} + m^2c_{44} + n^2c_{33} + 2mnc_{34} + 2nlc_{35} + 2lmc_{45}
\end{aligned} \tag{3f-4}$$

In these formulas c_{11} to c_{66} are the 21 elastic constants and l, m, n the direction cosines of the direction of propagation with respect to the crystallographic x, y, and z axes which are related to the a, b, c crystallographic axes as discussed in an IRE publication.[2]

In Eq. (3f-3), we solve for the quantity ρv^2. It was shown by Christoffel[2] that the direction cosines for the particle motion ξ, i.e., α, β, γ, are related to the λ constants and a solution of ρv^2 by the equations

$$\alpha\lambda_{11} + \beta\lambda_{12} + \gamma\lambda_{13} = \alpha\rho v_i^2 \quad \alpha\lambda_{12} + \beta\lambda_{22} + \gamma\lambda_{23} = \beta\rho v_i^2 \quad \alpha\lambda_{13} + \beta\lambda_{23} + \gamma\lambda_{33} = \gamma\rho v_i^2 \tag{3f-5}$$

where $i = 1, 2, 3$. Hence, solutions of Eq. (3f-3) are related to particle motions by the equations of (3f-5).

Most metals crystallize in the cubic and hexagonal systems. Furthermore, when a metal is produced by rolling, an alignment of grains occurs such that the rolling direction is a unique axis. This type of symmetry, known as transverse isotropy, results in the same set of constants as that for hexagonal symmetry. For cubic crystals, the resulting elastic constants are

$$c_{11} = c_{22} = c_{33} \qquad c_{12} = c_{13} = c_{23} \qquad c_{44} = c_{55} = c_{66} \tag{3f-6}$$

while for hexagonal symmetry or transverse isotropy, the resulting elastic constants are

$$c_{11} = c_{22} \qquad c_{12} \qquad c_{13} = c_{23} \qquad c_{44} = c_{55} \qquad c_{66} = \frac{c_{11} - c_{12}}{2} \tag{3f-7}$$

[1] Love, "Theory of Elasticity," 4th ed., p. 298, Cambridge University Press, New York, 1934.

[2] Standards on Piezoelectric Crystals, *Proc. IRE* **37** (12), 1378–1395 (December, 1949).

For cubic symmetry, the waves transmitted along the [100] direction and the [110] direction have purely longitudinal and shear components with the elastic-constant values and particle direction ξ given by

[100] direction

$$v_{\text{long}} = \sqrt{\frac{c_{11}}{\rho}} \quad \xi \text{ along } [100] \qquad v_{\text{shear}} = \sqrt{\frac{c_{44}}{\rho}}$$

ξ along any direction in the [100] plane

[110] direction

$$v_{\text{long}} = \sqrt{\frac{c_{11} + c_{12} + 2c_{44}}{2\rho}} \quad \xi \text{ along } [110]$$

$$v_{1\,\text{shear}} = \sqrt{\frac{c_{44}}{\rho}} \quad \xi \text{ along } [001] \qquad v_{2\,\text{shear}} = \sqrt{\frac{c_{11} - c_{12}}{2\rho}}$$

ξ along $[1\bar{1}0]$

For hexagonal or transverse isotropy, waves transmitted along the unique axis and any axis perpendicular to this will have the values

[001] direction

$$v_{\text{long}} = \sqrt{\frac{c_{33}}{\rho}} \quad \xi \text{ along } [001] \qquad v_{\text{shear}} = \sqrt{\frac{c_{44}}{\rho}}$$

ξ along any direction in the [001] plane

[100] direction

$$v_{\text{long}} = \sqrt{\frac{c_{11}}{\rho}} \quad \xi \text{ along } [100] \qquad v_{1\,\text{shear}} = \sqrt{\frac{c_{44}}{\rho}}$$

$$\xi \text{ along } [001] \qquad v_{2\,\text{shear}} = \sqrt{\frac{c_{11} - c_{12}}{2\rho}} \qquad \xi \text{ along } [010]$$

The fifth constant is measured by transmitting a wave 45 deg between the [100] and [001] directions, i.e., $l = n = 1/\sqrt{2}$; $m = 0$. For this case

$$\lambda_{11} = \frac{c_{11} + c_{44}}{2} \qquad \lambda_{12} = \lambda_{23} = 0 \qquad \lambda_{13} = \frac{c_{13} + c_{44}}{2} \qquad \lambda_{22} = \frac{c_{11} - c_{12} + 2c_{44}}{4}$$

$$\lambda_{33} = \frac{c_{44} + c_{33}}{2} \tag{3f-8}$$

The three solutions of Eq. (3f-3) are

$$\rho v_1^2 = \frac{c_{11} - c_{12} + 2c_{44}}{4}$$

$$\rho v_{2,3}^2 = \frac{[(c_{11} + c_{33} + 2c_{44})/2] \pm \sqrt{[(c_{11} - c_{33})/2]^2 + (c_{13} + c_{44})^2}}{2} \tag{3f-9}$$

For these three velocities, the particle velocities have the direction cosines

For v_1, $\beta = 1$

For v_2, $$\alpha = \gamma \left\{ \frac{c_{11} - c_{33}}{2(c_{13} + c_{44})} + \sqrt{1 + \left[\frac{c_{11} - c_{33}}{2(c_{13} + c_{44})} \right]} \right\} \tag{3f-10}$$

For v_3, $$\alpha = -\gamma \left\{ \frac{c_{33} - c_{11}}{2(c_{13} + c_{44})} + \sqrt{1 + \left[\frac{(c_{11} - c_{33})}{2(c_{13} + c_{44})}^2 \right]} \right\}$$

Hence, unless c_{11} is nearly equal to c_{33}, a longitudinal or shear crystal will generate both types of waves. Experimentally, however, it is found that a good discrimination can be obtained against the type of wave that is not primarily generated and a single velocity can be measured. A resonance technique can also be used to evaluate all the elastic constants of a crystalline material.

TABLE 3f-1. DENSITIES OF GLASSES, PLASTICS, AND METALS IN POLYCRYSTALLINE AND CRYSTALLINE FORM (X-RAY DENSITIES FOR CRYSTALS)*

Materials	Composition	Temp., °C	Density, kg/m³ × 10³ or g/cm³
Aluminum			
Hard-drawn	....	20	2.695
Crystal	....	25	2.697
Aluminum and copper	10 Al, 90 Cu	..	7.69
	5 Al, 95 Cu	..	8.37
	3 Al, 97 Cu	..	8.69
Beryllium	....	20	1.87
Crystal	....	18	1.871
Brass:			
Yellow	70 Cu, 30 Zn	..	8.5–8.7
Red	90 Cu, 10 Zn	..	8.6
White	50 Cu, 50 Zn	..	8.2
Bronze	90 Cu, 10 Sn	..	8.78
	85 Cu, 15 Sn	..	8.89
	80 Cu, 20 Sn	..	8.74
	75 Cu, 25 Sn	..	8.83
Chromium	....	20	6.92–7.1
Crystal	....	18	7.193
Cobalt	....	21	8.71
Crystal	....	..	8.788
Constantine	60 Cu, 40 Ni	..	8.88
Copper	....	..	8.3–8.93
Crystal	....	18	8.936
Duralumin	17ST = 4 Cu, 0.5 Mg, 0.5 Mn	..	2.79
Germanium	....	..	5.3
Crystal	....	20	5.322
German silver	26.3 Cu, 36.6 Zn, 36.8 Ni	..	8.30
	52 Cu, 26 Zn, 22 Ni	..	8.45
	59 Cu, 30 Zn, 11 Ni	..	8.34
	63 Cu, 30 Zn, 6 Ni	..	8.30
Gold	....	..	18.9–19.3
Crystal	....	20	19.32
Indium	....	..	7.28
Crystal	....	..	7.31
Invar	63.8 Fe, 36 Ni, 0.20 C	..	8.0
Iron	....	20	7.6–7.85
Crystal	....	20	7.87
Lead	....	20	11.36
Crystal	....	18	11.34
Lead and tin	87.5 Pb, 12.5 Sn	..	10.6
	84 Pb, 16 Sn	..	10.33
	72.8 Pb, 22.2 Sn	..	10.05
	63.7 Pb, 36.3 Sn	..	9.43
	46.7 Pb, 53.3 Sn	..	8.73
	30.5 Pb, 69.5 Sn	..	8.24

TABLE 3f-1. DENSITIES OF GLASSES, PLASTICS, AND METALS IN POLYCRYSTALLINE AND CRYSTALLINE FORM (X-RAY DENSITIES FOR CRYSTALS) (*Continued*)

Materials	Composition	Temp., °C	Density, kg/m³ × 10³ or g/cm³
Magnesium		..	1.74
Crystal		25	1.748
Manganese		..	7.42
Crystal		..	7.517
Mercury		20	13.546
Monel metal	71 Ni, 27 Cu, 2 Fe	..	8.90
Molybdenum		..	10.1
Crystal		25	10.19
Nickel		..	8.6–8.9
Crystal		25	8.905
Phosphor bronze	79.7 Cu, 10 Sn, 9.5 Sb, 0.8 P	..	8.8
Platinum		20	21.37
Crystal		18	21.62
Silicon		15	2.33
Crystal		25	2.332
Silver		..	10.4
Crystal		25	10.49
347 stainless steel		..	7.91
Tin		..	7–7.3
Crystal		..	7.3
Tungsten		..	18.6–19.1
Crystal		25	19.2
Zinc		..	7.04–7.18
Crystal		25	7.18
Fused silica		..	2.2
Pyrex glass (702)		..	2.32
Heavy silicate flint		..	3.879
Light borate crown		..	2.243
Lucite		..	1.182
Nylon 6-6		..	1.11
Polyethylene		..	0.90
Polystyrene		..	1.056

* See also Tables 26-1 through 26-13.

When a longitudinal or shear wave is reflected at an angle from a plane surface, both a longitudinal and a shear wave will in general be reflected from the surface, the angles of reflection and refraction satisfying Snell's law

$$\frac{\sin \beta}{v_S} = \frac{\sin \alpha}{v_l} \tag{3f-11}$$

where α and β are the angles of incidence and refraction with respect to a normal to the reflecting surface. Exceptions to this rule occur if a shear wave has its direction of particle displacement parallel to the reflecting surface, in which case only a pure shear wave is reflected, with the angle of reflection being equal to the angle of incidence. Use is made of this result in constructing delay lines which can be contained in a small volume. When the angle of incidence is 90 deg, the transmitted wave is reflected without change of mode. If the transmitting medium is connected to another medium with different properties, the transmission and reflection factors are determined by the relative impedances of the two media. The impedance is given by the formula

$$Z = \rho v = \sqrt{E\rho} \tag{3f-12}$$

where E is the appropriate elastic stiffness and ρ the density. The reflection and transmission coefficients between medium 1 and medium 2 are given by the equations

$$R = \frac{Z_1 - Z_2}{Z_1 + Z_2} \qquad T = 1 - R = \frac{2Z_2}{Z_1 + Z_2} \tag{3f-13}$$

Tables 3f-1 to 3f-4 list the densities, elastic constants, velocities, and impedances for a number of materials used in acoustic-wave propagation.

3f-2. Attenuation Due to Thermal Effects, Relaxations, and Scattering. When sound is propagated through a solid, it suffers a conversion of mechanical energy into heat. While all the causes of conversion are not known, a number of them are, and tables for these effects are given in this section.

3f-3. Loss Due to Heat Flow. When a sound wave is sent through a body, a compression or rarefaction occurs which heats or cools the body. This heat causes thermal expansions which alter slightly the elastic constants of the material. Since the compressions and rarefactions occur very rapidly, there is not time for much heat to flow and the elastic constants measured by sound propagation are the adiabatic constants. For an isotropic material, the adiabatic constants are related to the isothermal constants by the formulas[1]

$$\lambda^\sigma = \lambda^\theta + \frac{9\alpha^2 B^2 \Theta}{\rho C_v} \qquad \mu^\sigma = \mu^\theta \qquad Y_0{}^\sigma = Y_0{}^\theta + \left(\frac{\mu}{\lambda + \mu}\right)^2 \frac{9\alpha^2 B^2 \Theta}{\rho C_v} \tag{3f-14}$$

where the superscripts σ and θ indicate adiabatic and isothermal constants, α is the linear temperature coefficient of expansion, B the bulk modulus ($B = \lambda + \frac{2}{3}\mu$), Θ the absolute temperature in degrees Kelvin, ρ the density, and C_v the specific heat at constant volume. Table 3f-5 shows these quantities for a number of materials.

The difference between λ^σ and λ^θ should be taken account of when one compares the elastic constants measured by ultrasonic means with those measured by static means. From the data given in Table 3f-5, it is evident that this effect can produce errors as high as 10 per cent in the case of zinc. Adiabatic elastic constants are measured from frequencies somewhat greater than those for which thermal equilibrium is established during the cycle to a frequency[1] $f \doteq (\rho C_v v^2/2\pi K)$ for which wave propagation again takes place isothermally. This frequency is approximately 10^{12} cycles for most metals.

When account is taken of the energy lost by heat flow between the hot and cool parts, this adds an attenuation for longitudinal waves equal to

$$A = \frac{2\pi f^2}{\rho v^3} \left[\frac{K}{C_v}\left(\frac{E^\sigma - E^\theta}{E^\Theta}\right)\right] \quad \text{nepers/m} \tag{3f-15}$$

where f is the frequency, v the velocity, K the heat conductivity, and E the appropriate elastic constant for the mode of propagation considered. Since $Q = B/2A$, it becomes

$$Q = \frac{\rho C_v v^2}{2fK[(E^\sigma - E^\theta)/E^\Theta]} \tag{3f-16}$$

[1] W. P. Mason, "Piezoelectric Crystals and Their Application to Ultrasonics," pp. 480–481, D. Van Nostrand Company, Inc., New York, 1950.

TABLE 3f-2. ELASTIC CONSTANTS, WAVE VELOCITIES, AND CHARACTERISTIC IMPEDANCES OF METALS, GLASSES, AND PLASTICS

Materials	Y_0, newton/m² × 10^{-10}	μ, newton/m² × 10^{-10}	λ, newton/m² × 10^{-10}	Poisson's ratio, σ	$V_l = \sqrt{(\lambda + 2\mu)/\rho}$, m/sec	$V_s = \sqrt{\mu/\rho}$, m/sec	$V_{ext} = \sqrt{Y_0/\rho}$ m/sec	$Z_l = \sqrt{\rho(\lambda + 2\mu)}$, kg/sec m² × 10^{-6}	$Z_s = \sqrt{\rho\mu}$, kg/sec m² × 10^{-6}
Aluminum, rolled	6.8–7.1	2.4–2.6	6.1	0.355	6,420	3,040	5,000	17.3	8.2
Beryllium	30.8	14.7	1.6	0.05	12,890	8,880	12,870	24.1	16.6
Brass, yellow, 70 Cu, 30 Zn	10.4	3.8	11.3	0.374	4,700	2,110	3,480	40.6	18.3
Copper, rolled	12.1–12.8	4.6	13.1	0.37	5,010	2,270	3,750	44.6	20.2
Duralumin 17S	7.15	2.67	5.44	0.335	6,320	3,130	5,150	17.1	8.5
Gold, hard-drawn	8.12	2.85	15.0	0.42	3,240	1,200	2,030	62.5	23.2
Iron electrolytic	20.6	8.2	11.3	0.29	5,950	3,240	5,120	46.4	25.3
Armco	21.2	8.24	11.35	0.29	5,960	3,240	5,200	46.5	25.3
Lead, rolled	1.5–1.7	0.54	3.3	0.43	1,960	690	1,210	22.4	7.85
Magnesium, drawn, annealed	4.24	1.62	2.56	0.306	5,770	3,050	4,940	10.0	5.3
Monel metal	16.5–18	6.18–6.86	12.4	0.327	5,350	2,720	4,400	47.5	24.2
Nickel	21.4	8.0	16.4	0.336	6,040	3,000	4,900	53.5	26.6
Platinum	16.7	6.4	9.9	0.303	3,260	1,730	2,800	69.7	37.0
Silver	7.5	2.7	8.55	0.38	3,650	1,610	2,680	38.0	16.7
347 Stainless steel	19.6	7.57	11.3	0.30	5,790	3,100	5,000	45.7	24.5
Tin, rolled	5.5	2.08	4.04	0.34	3,320	1,670	2,730	24.6	11.8
Tungsten, drawn	36.2	13.4	31.3	0.35	5,410	2,640	4,320	103	50.5
Zinc, rolled	10.5	4.2	4.2	0.25	4,210	2,440	3,850	30	17.3
Fused silica	7.29	3.12	1.61	0.17	5,968	3,764	5,760	13.1	8.29
Pyrex glass	6.2	2.5	2.3	0.24	5,640	3,280	5,170	13.1	7.6
Heavy silicate flint	5.35	2.18	1.77	0.224	3,980	2,380	3,720	15.4	9.22
Light borate crown	4.61	1.81	2.2	0.274	5,100	2,840	4,540	11.4	6.35
Lucite	0.40	0.143	0.562	0.4	2,680	1,100	1,840	3.16	1.3
Nylon 6-6	0.355	0.122	0.511	0.4	2,620	1,070	1,800	2.86	1.18
Polyethylene	0.076	0.026	0.288	0.458	1,950	540	920	1.75	0.48
Polystyrene	0.528	0.12	0.34	0.405	2,350	1,120	2,240	2.48	1.18

TABLE 3f-3. ELASTIC CONSTANTS OF CUBIC SINGLE CRYSTALS*
(s = compliance modulus, m^2/newton; c = stiffness modulus, newtons/m^2; for cgs units of dynes/cm^2, multiply the c constants by 10; divide the s constants by 10 to obtain cm^2/dyne)

Crystal	$s_{11} \times 10^{11}$	$s_{12} \times 10^{11}$	$s_{44} \times 10^{11}$	$c_{11} \times 10^{-10}$	$c_{12} \times 10^{-10}$	$c_{44} \times 10^{-10}$	$B = [(c_{11} + 2c_{12})/3] \times 10^{-10}$	Anisotropy $2c_{44}/(c_{11} - c_{12})$
Ag	2.32	−0.993	2.29	11.9	8.94	4.37	9.93	2.95
Al	1.59	−0.58	3.52	10.82	6.13	2.85	7.69	1.24
Au	2.33	−1.07	2.38	19.6	16.45	4.20	17.5	2.67
Cu	1.49	−0.625	1.33	17.02	12.3	7.51	13.9	3.18
Fe	0.757	−0.282	0.862	23.7	14.1	11.6	17.3	2.37
Ge	0.964	−0.260	1.49	12.92	4.79	6.70	7.50	1.65
K	83.3	−37.0	38.0	0.416	0.333	0.263	0.361	6.34
Na	48.3	−20.9	16.85	0.615	0.469	0.592	0.518	8.11
Ni (sat.)	0.80	−0.312	0.844	25.0	16.0	11.85	19.0	2.63
Pb	9.30	−4.26	6.94	4.85	4.09	1.44	4.34	3.79
Si	0.768	−0.214	1.26	16.57	6.39	7.956	9.783	1.56
W	0.257	−0.073	0.66	50.2	19.9	15.15	30.0	1.0
Diamond†	0.0958	−0.01	0.174	107.6	12.5	57.6	44.2	1.21
NaCl	2.4	−0.50	7.8	0.49	0.124	0.126	0.25	0.688
KBr	4.0	−1.2	7.5	0.35	0.058	0.050	0.16	0.342
KCl	2.7	−0.3	15.6	0.40	0.062	0.062	0.17	0.361

Elastic Constants of Copper Alloys‡

Alloy	Atom % of second component	$s_{11} \times 10^{11}$	$s_{12} \times 10^{11}$	$s_{44} \times 10^{11}$	$c_{11} \times 10^{-10}$	$c_{12} \times 10^{-10}$	$c_{44} \times 10^{-10}$	$B = [(c_{11} + 2c_{12})/3] \times 10^{-10}$	Anisotropy $2c_{44}/(c_{11} - c_{12})$
CuZn	4.53	1.59	−0.671	1.348	16.34	11.92	7.42	13.39	3.36
CuAl	4.81	1.59	−0.674	1.335	16.58	12.16	7.49	13.63	3.39
	9.98	1.67	−0.711	1.305	15.95	11.77	7.66	13.16	3.66
CuGa	1.58	1.55	−0.65	1.346	16.49	11.93	7.43	13.45	3.25
	4.15	1.59	−0.672	1.349	16.51	12.10	7.41	13.57	3.36
CuSi	4.17	1.61	−0.685	1.336	16.78	12.42	7.48	13.87	3.43
	5.16	1.67	−0.709	1.335	16.09	11.88	7.49	13.28	3.56
	7.69	1.73	−0.745	1.350	16.64	12.60	7.41	13.95	3.67
CuGe	1.03	1.52	−0.637	1.333	16.66	12.00	7.50	13.62	3.29
	1.71	1.57	−0.663	1.333	16.30	11.83	7.50	13.32	3.35

* See also Tables 2f-1 through 2f-5.
† Recent data by W. L. Bond and H. J. McSkimin.
‡ Data from C. S. Smith.

TABLE 3f-4. ELASTIC CONSTANTS OF HEXAGONAL CRYSTALS
(s = compliance moduli, m^2/newton; c = stiffness moduli, newtons/m^2; for cgs units of dynes/cm^2 multiply the c constants by 10; divide the s constants by 10 to obtain cm^2/dyne)

Crystal	$s_{11} \times 10^{11}$	$s_{12} \times 10^{11}$	$s_{13} \times 10^{11}$	$s_{33} \times 10^{11}$	$s_{44} \times 10^{11}$	
Cd	1.23	−0.15	−0.93	3.55	5.40	
Mg	2.21	−0.77	−0.49	1.97	6.03	
Zn	0.84	+0.11	−0.78	2.87	2.64	
Co	0.473	−0.231	−0.07	0.319	1.325	
	$c_{11} \times 10^{-10}$	$c_{12} \times 10^{-10}$	$c_{13} \times 10^{-10}$	$c_{33} \times 10^{-10}$	$c_{44} \times 10^{-10}$	$B = \dfrac{1}{2(s_{11} + s_{12}) + s_{33} + 4s_{13}} \times 10^{-10}$
Cd	12.12	4.81	4.42	4.45	1.85	5.03
Mg	5.86	2.49	2.08	6.60	1.65	3.46
Zn	16.35	2.64	5.17	5.31	3.78	8.26
Co	30.71	16.5	10.27	35.81	7.55	19.01

TABLE 3f-5. ADIABATIC ISOTHERMAL ELASTIC CONSTANTS AND ATTENUATION DUE TO HEAT FLOW

Material	Density, kg/m^3 × 10^{-3}	C_v, joules/kg/°C × 10^{-3}	α, × 10^6/°C	K, watts/m^2/m/°C × 10^{-2}	λ^θ, newtons/m^2 × 10^{-10}	μ, newtons/m^2 × 10^{-10}	$\lambda^\sigma - \lambda^\theta$, newtons/m^2 × 10^{-9}	$Y_0^\sigma - Y_0^\theta$, newtons/m^2 × 10^{-8}	A/f^2, nepers/m
Aluminum	2.699	0.9	23.9	2.22	6.1	2.5	3.8	3.2	2.3×10^{-16}
Beryllium	1.82	2.17	12.4	1.58	1.6	14.7	1.4	11.4	2.1×10^{-18}
Copper	8.96	0.384	16.5	3.93	13.1	4.6	5.5	3.7	4.45×10^{-16}
Gold	19.32	0.13	14.2	2.97	15.0	2.85	6.1	1.5	1.95×10^{-15}
Iron	7.87	0.46	11.7	0.75	11.3	8.2	2.7	4.8	1.88×10^{-17}
Lead	11.4	0.128	29.4	0.344	3.3	0.54	2.12	0.36	2.95×10^{-15}
Magnesium	1.74	1.04	26	1.59	2.56	1.62	1.3	2.1	2.0×10^{-16}
Nickel	8.90	0.44	13.3	0.92	16.4	8.0	5.7	6.1	3.8×10^{-17}
Silver	10.49	0.234	19.7	4.18	8.55	2.7	4.5	2.6	1.95×10^{-15}
Tin	7.3	0.225	23	0.67	4.04	2.08	3.5	4.0	9.7×10^{-16}
Tungsten	19.3	0.134	4.3	2.0	31.3	13.4	3.1	2.8	5.0×10^{-17}
Zinc	7.1	0.382	29.7	1.12	4.2	4.2	4.3	10.7	3.8×10^{-16}
Fused silica	2.2	0.92	0.5	0.01	1.61	3.12	0.00045	0.002	2.6×10^{-22}

TABLE 3f-6. FACTORS GOVERNING INTERGRAIN HEAT FLOW IN METALS

Metal	Pb	Ag	Cu	Au	Fe	Al	W
R	0.065	0.031	0.031	0.014	0.022	0.0009	10^{-1}
$(C_p - C_v)/C_v$	0.067	0.040	0.028	0.038	0.016	0.046	0.006
Product	4.4×10^{-3}	1.2×10^{-3}	8.7×10^{-4}	5.3×10^{-4}	3.5×10^{-4}	4×10^{-5}	6×10^{-9}

TABLE 3f-7. RELATIVE SCATTERING FACTORS FOR LONGITUDINAL AND SHEAR WAVES IN POLYCRYSTALLINE METALS

Metal	Al	Au	Ag	Cu	Pb	Fe	Na	K	W	Mg	Zn	Cd
S_L	3×10^{-4}	1.78×10^{-3}	5×10^{-3}	7.4×10^{-3}	4.2×10^{-3}	6.7×10^{-3}	2.9×10^{-2}	1.7×10^{-2}	0	2.2×10^{-4}	5.6×10^{-2}	2.8×10^{-2}
S_S	3.3×10^{-3}	5.2×10^{-2}	6.1×10^{-2}	6.7×10^{-2}	7.2×10^{-2}	4.0×10^{-2}	1.25×10^{-1}	1.1×10^{-1}	0			

where Q is the ratio of 2π times the energy stored to energy dissipated per cycle and B is the phase shift per unit length. Table 3f-5 shows the attenuation for a number of solids due to thermal loss.

3f-4. Loss Due to Intergrain Heat Flow. A related thermal loss that occurs in polycrystalline material is the thermoelastic relaxation loss which arises from heat flow from grains that have received more compression or extension in the course of the wave motion than do adjacent grains. The Q from this source has been shown to be[1]

$$\frac{1}{Q} = \frac{C_p - C_v}{C_v} R \frac{f_0 f}{f_0^2 + f^2} \tag{3f-17}$$

where R is that fraction of the total strain energy which is associated with the fluctuations of dilations, and f_0, the relaxation frequency, is approximately

$$f_0 = \frac{D}{L_c^2} = \frac{K}{\rho C_p L_c^2} \tag{3f-18}$$

where L_c is the mean diameter of the crystallites and D the diffusion constant.

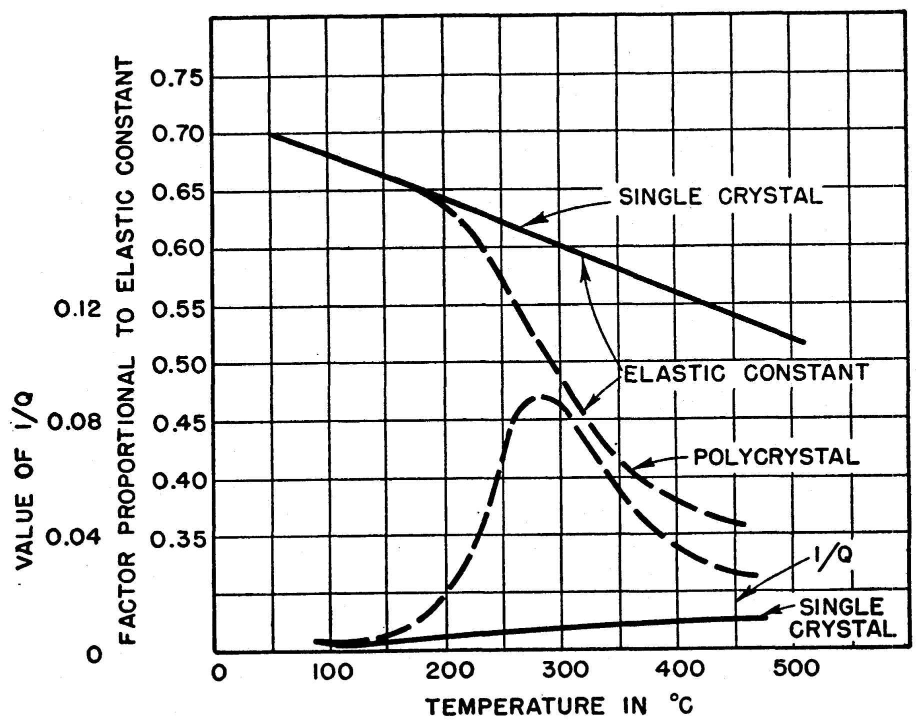

FIG. 3f-1. Elastic constants and Q for single-crystal and polycrystal aluminum. (*After Kê.*)

For most materials, the relaxation frequencies are under 100 kc. Table 3f-6 gives the product $[(C_p - C_v)/C_v]R$ for a number of metals.

3f-5. Loss Due to Grain Rotation. Another source of loss due to grain structure in metals is the loss due to the viscosity of the boundary layer between grains. This allows a relative rotation of grains provided the relaxation time is comparable with the time of the applied force. Figure 3f-1 shows the elastic modulus and the associated Q of a polycrystalline aluminum rod in torsional vibration at a frequency of 0.8 cycle as compared with similar measurements for a single crystal. The relaxation time for grain-boundary rotation is a function of temperature according to the equation

$$\tau = \tau_0 e^{H/kT} \tag{3f-19}$$

[1] C. Zener, "Elasticity and Anelasticity of Metals," p. 84, University of Chicago Press, Chicago, 1948.

where H, the activation energy, is of the same order as that found for creep and self-diffusion.

3f-6. Loss Due to Grain Scattering of Sound. Another effect of grain structure in solids is a loss of energy from the main wave due to the scattering of sound when the sound wavelength is of the same order as the grain size. This scattering occurs because adjacent grains have different orientations, and a reflection of sound occurs because of the resulting impedance difference between grains. An approximate formula[1] holding when the wavelength is larger than three times the grain size, and multiple scattering is neglected, is

$$\alpha_s \doteq \frac{8\pi^4 L_c^3 f^4}{9v^4} S \qquad \text{nepers/m} \tag{3f-20}$$

where L_c is the average grain diameter, f the frequency, v the velocity, and S a scattering factor related to the anisotropy of the metal. Table 3f-7 shows a relative estimate of the scattering factors for longitudinal and shear waves for a number of metals. For shorter wavelengths, the attenuation changes less rapidly with frequency,[2] and for wavelengths shorter than the grain size, the loss is independent of the frequency. A formula applicable for all wavelengths is

$$\alpha_s = \frac{S}{2L_c}\left(\frac{Q_s}{A}\right) \tag{3f-21}$$

where Q_S/A is the ratio of the scattering area of the sphere to the actual cross-sectional area. For low frequencies $Q_S/A = \frac{16}{9}(\pi L_c/\lambda)^4 = \frac{16}{9}(\pi L_c f/v)^4$ while for very high frequencies $Q_S/A = 2$. Intermediate values of cross-sectional areas can be obtained from calculations given by Morse.[3] Because of elongations of grains in the direction of rolling, most materials have different scattering areas for propagation along the rolling axis and perpendicular to the axis.

3f-7. Acoustic Losses in Ferromagnetic and Ferroelectric Materials. Stresses in ferromagnetic and ferroelectric materials can cause motion of domain walls or rotation of domain directions. These occur in such a manner that domains are strengthened in directions parallel, antiparallel, or perpendicular to the direction of the stress. The increased polarization in the direction of the stress produces increased strains which are the same sign in both parallel and antiparallel domains since magnetostriction and electrostriction are square-law effects and hence the elastic stiffnesses of demagnetized materials are less than those of completely magnetized materials. For polarizations directed along cube axes, the difference in elastic constants for the saturated and depolarized states, i.e., the ΔE effect, is[4]

$$\frac{\Delta E}{E_D} = \frac{9\mu\lambda_s^2 E_s}{20\pi P_s^2} \tag{3f-22}$$

where μ is the initial permeability or dielectric constant, λ_s the saturated change in length along a polycrystalline rod, E_s and E_D the saturated and demagnetized elastic-stiffness constant and P_s the saturated magnetic or electric polarization. When the polarization lies along a cube diagonal—as in nickel—λ_s is replaced by $\frac{2}{3}\lambda_{111}[5c_{44}/(c_{11} - c_{12} + 3c_{44})]$ where λ_{111} is the saturated increase in length along the [111] direction and $5c_{44}/(c_{11} - c_{12} + 3c_{44})$ is a ratio of elastic constants.

[1] Mason, *op. cit.*, p. 422.

[2] R. B. Roney, "The Influence of Metal Grain Structure on the Attenuation of Ultrasonic Waves," Thesis, California Institute of Technology, 1950.

[3] Philip M. Morse, "Vibration and Sound," 2d ed., p. 355, McGraw-Hill Book Company, Inc., New York, 1948.

[4] R. M. Bozorth, "Ferromagnetism," p. 691, D. Van Nostrand Company, Inc., New York, 1951.

The motion of walls or the rotation of domains in metallic ferromagnetic materials generates eddy currents and hence causes an acoustic loss. It has been shown that the permeability follows a relaxation equation

$$\mu = \mu_0 \frac{(1 - jf/f_0)}{1 + f^2/f_0^2} \tag{3f-23}$$

where $f_0 \doteq 4R/25\mu_0 L_c^2$, R = resistivity, and L_c = domain diameter. For a distribution of domain sizes

$$\mu = \mu_0 \sum_{i=1}^{m} \frac{V_i}{V} \frac{1 - jf/f_i}{1 + f^2/f_i^2} \tag{3f-24}$$

where V_i is the volume occupied by domains of size L_i and V the total volume.

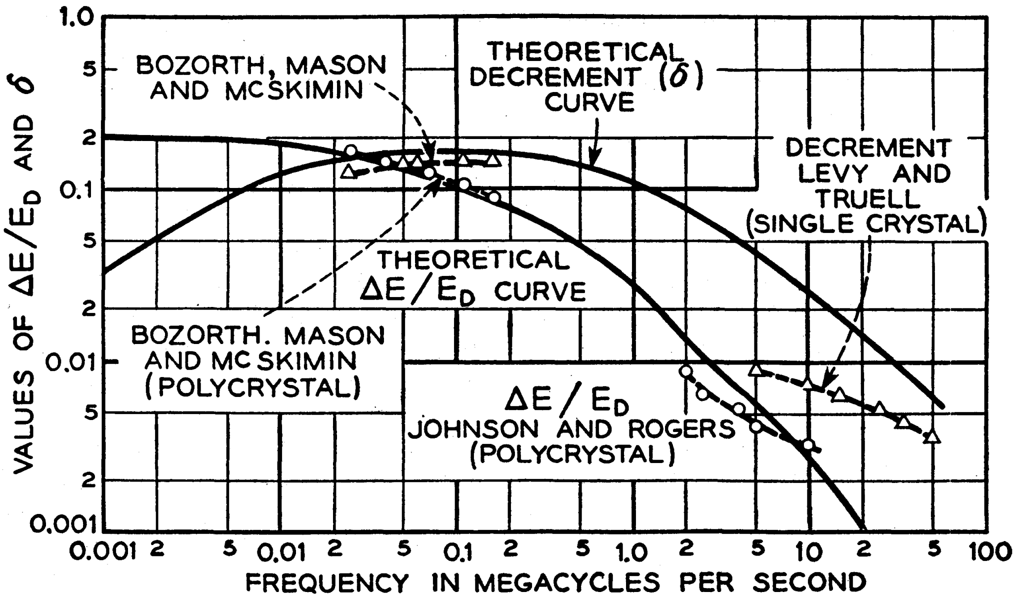

FIG. 3f-2. Decrement and ΔE effect for polycrystal nickel rod as a function of frequency. (*After Bozorth, Mason, and McSkimin; Johnson rnd Rogers; and Levy and Truell.*)

Inserting in Eq. (3f-22) the $\Delta E/E_D$ and Q are given by

$$\frac{\Delta E}{E_D} = \frac{9\lambda_s^2 E_s}{20\pi P_s^2}\left(\sum_{i=1}^{m} \frac{V_i/V}{1 + f^2/f_i^2}\right) \qquad \frac{1}{Q} = \frac{9\lambda_s^2 E_s}{20\pi P_s^2}\left[\sum \frac{(V_i/V)(f/f_i)}{1 + (f/f_i)^2}\right] \tag{3f-25}$$

Figure 3f-2 shows measurements of the ΔE effect and the decrement $\delta = \pi/Q$ plotted over a frequency range, for a polycrystalline nickel rod.

Another effect causing losses in ferromagnetic and ferroelectric materials is the microhysteresis effect. In this effect the domain walls or domain rotations lag behind the applied stress and produce a hysteresis loop. Hence the initial susceptibility has a hysteresis component which is a function of the amount of stress. Average values of the parameters can be written in the form

$$\mu = \mu_0[1 - jf(A)] \tag{3f-26}$$

where $f(A)$ is a function of the amplitide. Inserting this value of μ in Eq. (3f-22), the value of the microhysteresis loss is given. This type of loss is present in ferroelectric materials and is the principal cause of the low mechanical Q.

3f-8. Other Types of Losses. In addition to these recognized types of losses, other types exist which have not been accounted for quantitatively. Figure 3f-3 shows the Q of a number of materials measured in a frequency range for strains under 10^{-5}.[1] Except for nickel and iron rods whose decrease in Q with frequency is accounted for by microeddy-current effects, the materials have a Q independent of frequency. It has

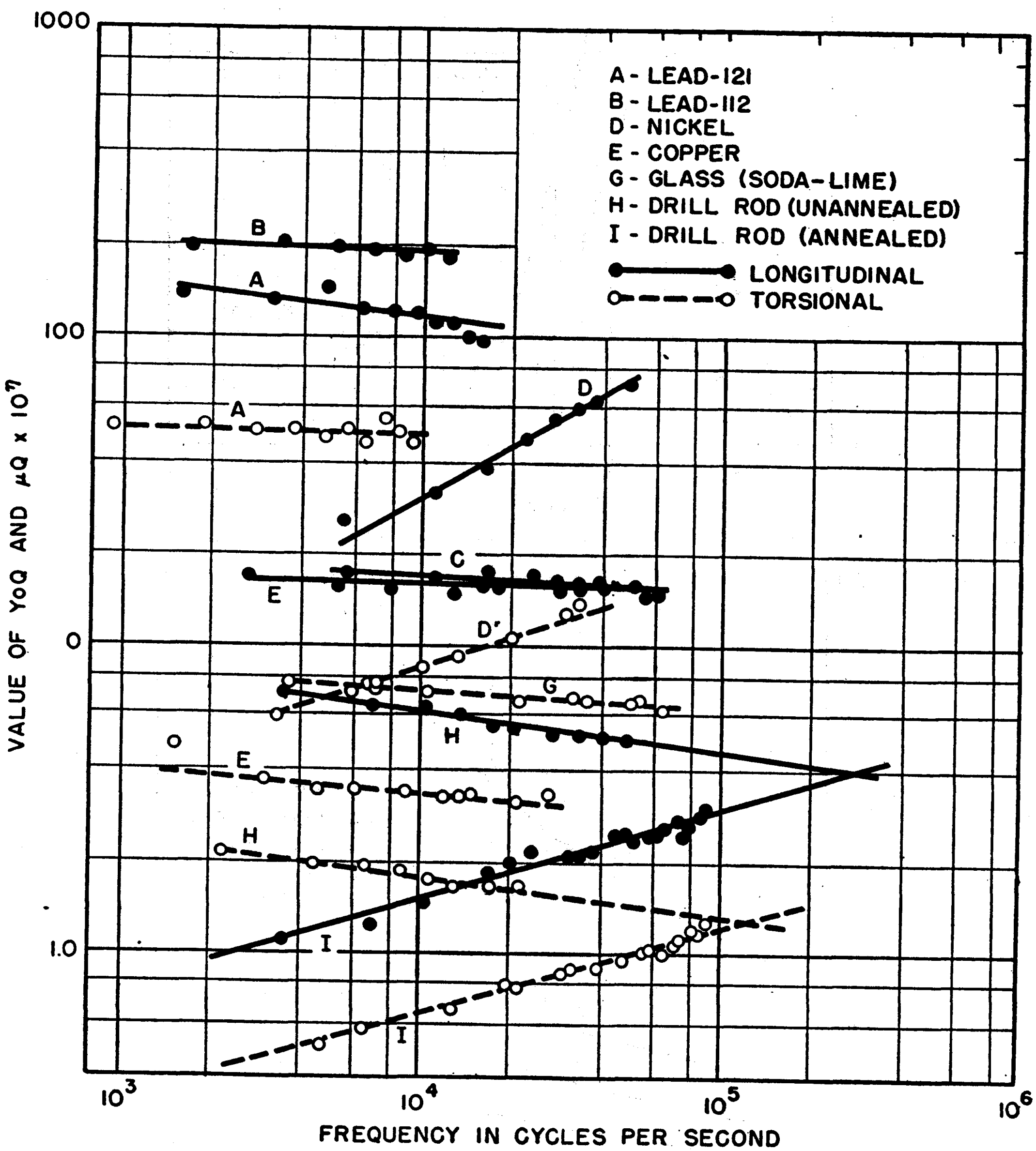

FIG. 3f-3. Values of Y_0/Q and μ/Q as a function of frequency for a number of polycrystalline materials. (*After Wegel and Walther.*)

been suggested that these losses are caused by elastic-hysteresis effects due to cyclic displacements of dislocations in the body or grain boundaries of metals. Some evidence[2] for this is shown in Fig. 3f-4, which shows the Q of a copper rod as a function of temperature and degree of annealing. Losses in annealed specimens having smaller numbers of dislocations are smaller than those in cold-worked specimens. At low

[1] R. L. Wegel and H. Walter, *Physics* **6**, 141 (1935).

[2] P. G. Bordoni, Assorbimento degli Ultrasuoni nei solidi, *Nuovo cimento* **7** (2), 144 (1950).

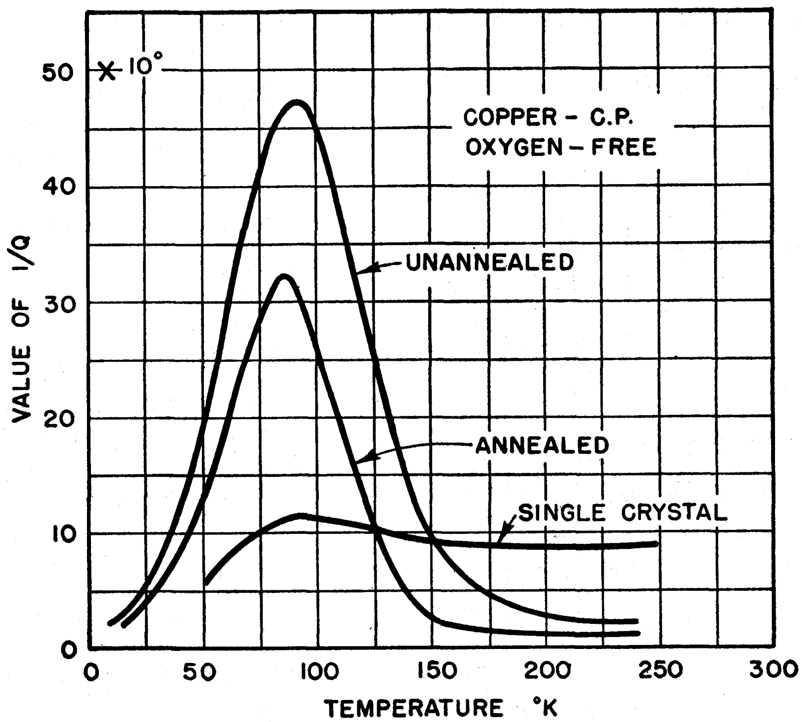

FIG. 3f-4. Attenuation peak in polycrystalline and single-crystal copper. (*After Bordoni.*)

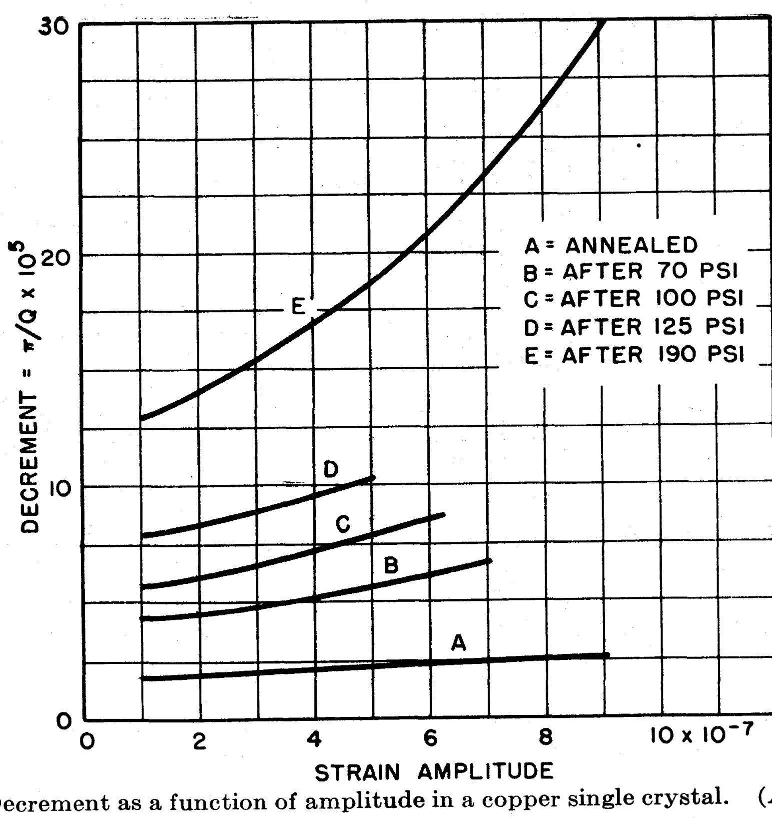

FIG. 3f-5. Decrement as a function of amplitude in a copper single crystal. (*After Nowick.*)

temperatures, a relaxation of dislocation motions appears to occur. Other work[1] shows that losses increase as a function of the amplitude, as shown by Fig. 3f-5. These losses have an activation energy similar to that shown by Fig. 3f-4 and are increased by cold work.

[1] A. S. Nowick, *Phys. Rev.* **80,** 249 (1950).

3g. Properties of Transducer Materials

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To determine the acoustic properties of gases, liquids, and solids and to utilize them in acoustic systems, it is necessary to generate the appropriate waves by means of transducer materials which convert electrical energy into mechanical energy and vice versa. For liquids and solids, the most common types of materials are piezoelectric crystals, ferroelectric materials of the barium titanate type, and magnetostrictive materials.

3g-1. Piezoelectric Crystals. The static relations for a piezoelectric quartz crystal producing a single longitudinal mode are for rationalized mks units

$$S_2 = s_{22}{}^E T_2 + d_{21} E_x \qquad D_x = d_{21} T_2 + \epsilon_1{}^T E_x \tag{3g-1}$$

where S_2 and T_2 are the longitudinal strain and stress, respectively, $s_{22}{}^E$ the elastic compliance along the length measured at constant electric field, d_{21} the piezoelectric constant relating the strain with the applied field E_x, D_x the electric displacement, and $\epsilon_1{}^T$ the dielectric constant measured at constant stress. Equations of this type suffice to determine the static and low-frequency behavior of piezoelectric crystals. Using the first equation, one finds that the increase in length for no external stress and the external force for no increase in length are, respectively,

$$\Delta l = d_{21} \frac{Vl}{t}; \qquad F = T_2 t w = -d_{21} \frac{Vw}{s_{22}{}^E} \tag{3g-2}$$

where V is the applied potential, l, w, and t are the length, width, and thickness of the crystal, and F is the force which is considered positive for an extensional stress. From the second equation one finds that the open-circuit voltage and the short-circuited charge for a given applied force are, respectively,

$$V = -\left(\frac{d_{21}}{\epsilon^T}\right) \frac{lF}{tw} \qquad Q = \int_0^l \int_0^w D_x \, dl \, dw = d_{21} \frac{Fl}{t} \tag{3g-3}$$

Another important criterion for transducer use is the electromechanical-coupling factor k whose square is defined as the ratio of the energy stored in mechanical form to the total input electrical energy. Using Eqs. (3g-1), this can be shown to be

$$k^2 = \frac{d_{21}{}^2}{s_{22}{}^E \epsilon^T} \tag{3g-4}$$

It is readily shown that the clamped dielectric constant ϵ^S, obtained by setting $S_2 = 0$, and the constant-displacement elastic compliance s^D, obtained by setting $D_x = 0$, are related to the constant-stress dielectric constant ϵ^T and the constant-field elastic compliance $s_{22}{}^E$ by the equations

$$\frac{\epsilon_1{}^S}{\epsilon_1{}^T} = \frac{s_{22}{}^D}{s_{22}{}^E} = 1 - k^2 \tag{3g-5}$$

Equivalent circuits in which the properties of the crystal are expressed in terms of equivalent electrical elements are often useful (see Secs. 3l and 3m). An equivalent circuit for a piezoelectric crystal for static conditions is shown by Fig. 3g-1*A*. In this network the compliance $C_1 = s_{22}^E l/wt$ represents the compliance of the crystal with the electrodes short-circuited, the capacitance C_0 is the capacitance of the clamped crystal, i.e., $C_0 = lw\epsilon_1^S/t$, while the transformer shown is a perfect transformer, i.e., a transformer having no loss between zero frequency and the highest frequency for which the piezoelectric effect is operative, having a turns ratio of φ to 1 where

$$\varphi = -d_{21} \frac{w}{s_{22}^E} \tag{3g-6}$$

The fact that this equivalent circuit presents the same information as Eq. (3g-1) is readily verified by substitution and integration over the area of the crystal.

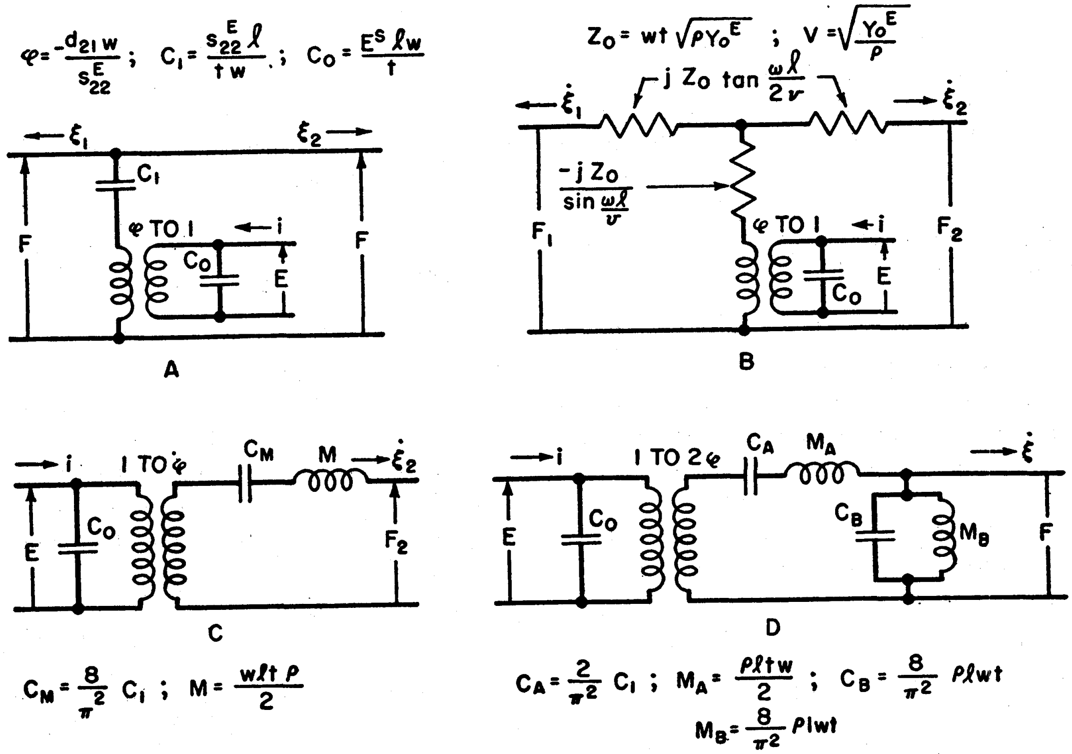


FIG. 3g-1. Equivalent circuit for a piezoelectric crystal for clamped and free conditions.

As an example of the use of such a network, one can calculate from it the efficiency of transformation of mechanical to electrical energy, or vice versa, under various conditions. Suppose that we clamp one end of the crystal and apply a force through the sending-end mechanical resistance R_M and receive the power generated into an electrical resistance R_E. Solving the network equations and obtaining the conditions for maximum power output, it is readily shown that the maximum power is obtained if

$$R_M = \frac{1}{\omega C_1 \sqrt{1 - k^2}} \qquad R_E = \frac{\sqrt{1 - k^2}}{\omega C_0} \tag{3g-7}$$

where $\omega = 2\pi$ times the frequency f. With these values the power in the termination is

$$P_0 = \frac{F^2 k^4}{4\varphi^2 R_E} \tag{3g-8}$$

The available power that can be obtained from a source having an open-circuit force F with an internal impedance R_M is maximum when $\varphi^2 R_E = R_M$. This power is then

$$i_2^2 R_E = \frac{F^2}{4\varphi^2 R_E} \tag{3g-9}$$

and hence the power-conversion efficiency is

$$P_E = k^4 \tag{3g-10}$$

Hence, unless the coupling is high, the efficiency of conversion by static means is low.

This efficiency can be improved by resonating the capacity C_0 by an electric coil L_0 at the frequency of operation and can be further improved by mechanically resonating the static compliance of the crystal. The simplest way to analyze these circuits for their optimum conditions is to observe that, if the perfect transformer is moved to the end of the circuit, both equivalent sections are half sections of well-known filters. Equation (3g-11) gives the element values of the first filter resonated by an electrical coil, while Eq. (3g-14) gives the element values for the section tuned on both ends.

$$\begin{aligned} C_1' &= \frac{s_{22}^E l}{wt}\left(\frac{d_{21}w}{s_{22}^E}\right)^2 = \frac{lw}{t}\frac{d_{21}^2}{s_{22}^E} = \frac{f_1+f_2}{2\pi f_1 f_2 Z_0'} = \frac{1}{4\pi f_1 Z_0}, \\ C_0 &= \frac{\epsilon^S lw}{t} = \frac{f_1}{2\pi f_2(f_2-f_1)Z_0'} = \frac{f_1}{2\pi(f_2^2-f_1^2)Z_0} \\ L_0 &= \frac{(f_2-f_1)Z_0'}{2\pi f_1 f_2} = \frac{(f_2^2-f_1^2)Z_0}{2\pi f_1 f_2^2} \end{aligned} \tag{3g-11}$$

where f_1 is the lower cutoff, f_2 the upper cutoff, Z_0 the mid-shunt impedance occurring on the electrical side, and Z_0' the mid-series impedance occurring on the mechanical side. Solving for f_1, f_2, Z_0, and $Z_0'(\varphi^2)$, i.e., the actual mechanical resistance, we find

$$f_2 = \frac{1}{2\pi\sqrt{L_0C_0}} \qquad f_1 = \frac{\sqrt{1-k^2}}{2\pi\sqrt{L_0C_0}} \qquad Z_0 = R_E = \frac{1-k^2}{2\pi f_1 C_0}$$

$$R_M = \varphi^2 Z_0' = \frac{1+\sqrt{1-k^2}}{2\pi f_1(l s_{22}^E/tw)} \tag{3g-12}$$

Hence, if there is no dissipation in the elements of the crystal, perfect power conversion can be obtained but only over a bandwidth of

$$\frac{f_2-f_1}{f_2} = 1 - \sqrt{1-k^2} \tag{3g-13}$$

The other section of Fig. 3g-2 is a wider bandpass filter having the element values

$$C_1' = \frac{lw}{t}\frac{d_{21}^2}{s_{22}^E} = \frac{f_2-f_1}{2\pi f_1 f_2 Z_0} \qquad L_1' = \frac{\rho lt}{w}\left(\frac{s_{22}^E}{d_{21}}\right)^2 = \frac{Z_0}{2\pi(f_2-f_1)}$$

$$C_0 = \frac{\epsilon^S lw}{t} = \frac{1}{2\pi(f_2-f_1)Z_0} \qquad L_0 = \frac{(f_2-f_1)Z_0}{2\pi f_1 f_2} \tag{3g-14}$$

Solving for the bandwidth and the impedances

$$\frac{f_2-f_1}{f_m} = \frac{k}{\sqrt{1-k^2}} \qquad f_m = \sqrt{f_1f_2} = \frac{1}{2\pi\sqrt{L_0C_0}} = \frac{1}{2\pi\sqrt{L_1C_1}}$$

$$Z_0 = R_E = \frac{\sqrt{1-k^2}}{2\pi f_m C_0 k} \qquad R_M = \varphi^2 Z_0 = \frac{k}{\sqrt{1-k^2}}\frac{1}{2\pi f_m s_{22}^E}\frac{wt}{l} \tag{3g-15}$$

This filter section can efficiently transform mechanical into electrical energy and vice versa with a loss determined only by the dissipation in the elements of the crystal.

The simplest method for mechanically resonating the crystal is to use it near its natural mechanical resonance. An exact equivalent circuit for a vibrating crystal is shown by Fig. 3g-1*B*. Near the first resonant frequency, the equivalent circuit for a clamped quarter-wave crystal is shown by Fig. 3g-1*C* while the equivalent circuit for a half-wave crystal is shown by Fig. 3g-1*D*. When the half-wave crystal resonated by a shunt coil is applied to converting electrical into mechanical energy, the same formulas given in Eqs. (3g-14) and (3g-15) and applicable except that $k^2/(1 - k^2)$ is replaced by $(8/\pi^2)[k^2/(1 - k^2)]$. By using the complete representation of Fig. 3g-1*B* the effect can be calculated by using various backing plates on the radiation from the front surface.

The general form of Eq. (3g-1) holds for any single mode whether it is longitudinal or transverse as long as the appropriate constants are used. For longitudinal thickness modes when the radiating surface is a number of wavelengths in diameter, $s_{22}{}^E$ is replaced by $1/c_{11}{}^E$ and d_{21} by $e_{21}/c_{11}{}^E$, the appropriate thickness piezoelectric constant. For a thickness shear mode, the appropriate shear stiffness (c_{44}, c_{55}, or c_{66})

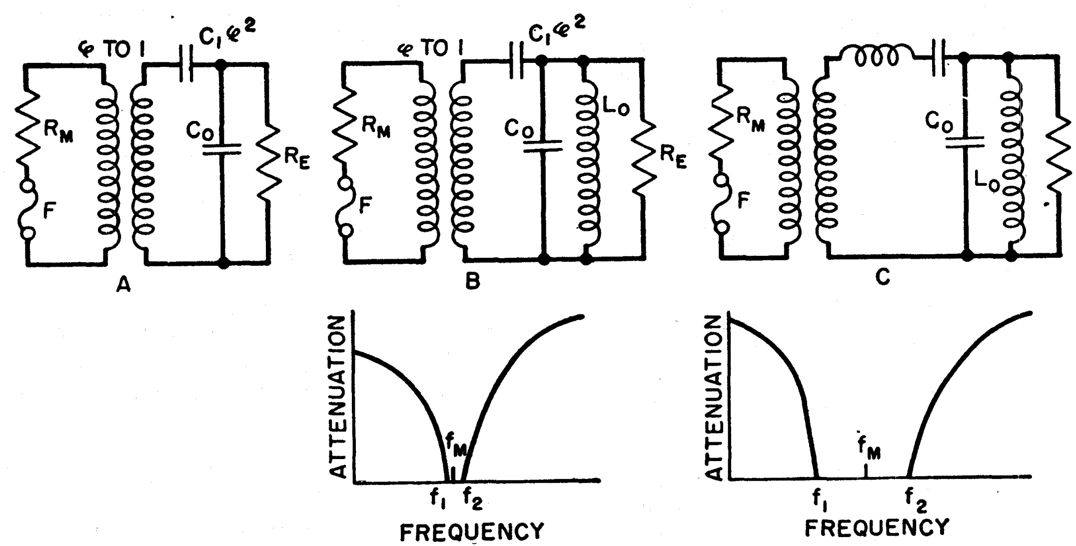


FIG. 3g-2. Use of equivalent circuit in determining the optimum conditions for energy transmission.

replaces $1/s_{22}$ and the appropriate shear piezoelectric constant replaces d_{21}. Table 3g-1 lists the constants in mks units for a number of standard crystal cuts.

3g-2. Electrostrictive and Magnetostrictive Materials. Other types of materials that have been used in transducers are ferroelectric crystals and ceramics of the barium titanate type and ferromagnetic crystals, polycrystals, and sintered materials of the ferrite type. All these materials have changes in lengths proportional to squares and even powers of the polarization and to obtain a linear response they have to be polarized. These polarized materials have relations between stresses, strains, electric and magnetic fields, and electric displacement and magnetic flux similar to those for a piezoelectric crystals shown by Eq. (3g-1) and hence these materials can be said to have "equivalent" constants which depend not only on the material but also on the degree of poling and in some cases on aging effects. The dielectric and permeability constants are those associated with the polarized medium as are also the elastic constants.

To obtain these equivalent piezoelectric and piezomagnetic constants, one can start with the more fundamental potential equations which have the same form for either electrostrictive or magnetostrictive materials. For polycrystalline or sintered materials, these potential equations can be written in the form

TABLE 3g-1. PROPERTIES OF PIEZOELECTRIC CRYSTALS IN MKS UNITS

Crystal and cut	Mode	Elastic constant, m^2/newton $\times 10^{11}$	Piezoelectric constant d, coulombs/newton $\times 10^{12}$	Dielectric capacitivity ϵ, farads/m $\times 10^{11}$	Electro-mechanical coupling k	Open-circuit voltage $g = d/\epsilon$, m^2/newton	Force factor d/s, newtons/volt $\times w$	Density, kg/m^3 $\times 10^{-3}$
Quartz *X* cut, length *Y*	L.L.	$s_{22}^E = 1.27$	$d_{21} = 2.25$	4.06	0.099	0.055	0.177	2.65
X cut	T.L.	$\frac{1}{c_{11}^E} = 1.16$	$\frac{e_{11}}{c_{11}^E} = -2.04$	4.06	0.093	0.050	0.175	2.65
Y cut	T.S.	$\frac{1}{c_{66}^E} = 2.57$	$\frac{e_{26}}{c_{66}^E} = +4.4$	4.06	0.137	0.108	0.171	2.65
Rochelle salt, 45-deg *X* cut	L.L.	$s'^E_{22} = 6.7$	$\frac{d_{14}}{2} = 435$	444.0	0.78	0.098	6.5	1.77
45-deg *Y* cut	L.L.	$s'^E_{11} = 9.89$	$\frac{d_{25}}{2} = -28.4$	9.85	0.288	0.29	0.287	1.77
ADP; 45-deg *Z* cut	L.L.	$s^{E'}_{11} = 5.3$	$\frac{d_{36}}{2} = 24.6$	13.8	0.29	0.178	0.465	1.804
KDP; 45-deg *Z* cut	L.L.	$s^{E'}_{11} = 4.85$	$\frac{d_{36}}{2} = 10.7$	19.6	0.12	0.058	0.22	2.31
EDT; *Y* cut, length *X*	L.L.	$s_{11}^E = 3.88$	$d_{21} = 11.3$	7.4	0.215	0.152	0.29	1.538
DKT; 45-deg *Z* cut	L.L.	$s^{E'}_{11} = 4.25$	$d'_{31} = -12.2$	5.8	0.245	0.21	0.287	1.988
L.H.; *Y* cut	T.L.	$\frac{1}{c_{22}^E} = 2$	$\frac{e_{22}}{c_{22}^E} = 15$	9.15	0.35	0.165	0.75	2.06
Hydrostatic	H.		$d_{21} + d_{22} + d_{23} = 13$	9.15		0.143		
Tourmaline, *Z* cut	T.L.	$\frac{1}{c_{33}^E} = 0.61$	$\frac{e_{33}}{c_{33}^E} = -1.84$	6.65	0.092	0.0275	0.3	3.1
Hydrostatic	H.		$d_{31} + d_{33} = -2.16$	6.65		0.0325		

Abbreviations: L.L. = length longitudinal; T.L. = thickness longitudinal; T.S. = thickness shear; ADP = ammonium dihydrogen phosphate; KDP = potassium dihydrogen phosphate; EDT = ethylene diamine tartrate; L.H. = lithium sulphate monohydrate.

$$\begin{aligned} G^* = & -\tfrac{1}{2}[s_{11}^D(T_1^2 + T_2^2 + T_3^2) + 2s_{12}^D(T_1T_2 + T_1T_3 + T_2T_3) \\ & + 2(s_{11}^D - s_{12}^D)(T_4^2 + T_5^2 + T_6^2)] - \{Q_{11}(D_1^2T_1 + D_2^2T_2 + D_3^2T_3) \\ & + Q_{12}[T_1(D_2^2 + D_3^2) + T_2(D_1^2 + D_3^2) + T_3(D_1^2 + D_2^2)] \\ & + 2(Q_{11} - Q_{12})(T_4D_2D_3 + T_5D_1D_3 + T_6D_1D_2)\} + \tfrac{1}{2}\beta_{11}^T(D_1^2 + D_2^2 + D_3^2) \\ & + K_{11}^T(D_1^4 + D_2^4 + D_3^4) + K_{12}^T(D_1^2D_2^2 + D_1^2D_3^2 + D_2^2D_3^2) \\ & + K_{111}^T(D_1^6 + D_2^6 + D_3^6) + K_{112}^T[D_1^4(D_2^2 + D_3^2) + D_2^4(D_1^2 + D_3^2) \\ & + D_3^4(D_1^2 + D_2^2)] + K_{123}^T D_1^2D_2^2D_3^2 \end{aligned} \tag{3g-16}$$

where T_1, T_2, T_3 are the three extensional stresses, T_4, T_5, T_6 the three shearing stresses, D_1, D_2, D_3 the three components of the electrical displacement for ferroelectric materials or the three components of the magnetic flux B for ferromagnetic materials, the s constants are the compliance constants for an isotropic material measured at constant electric or magnetic displacement, the Q's are the electrostrictive or magnetostrictive constants, β_{11}^T the inverse of the initial dielectric constant or permeability measured at constant stress, and the K^T's are constants determining the total energy stored for higher polarizations. The static equations can be obtained by differentiation of G according to the relations

$$S_i = -\frac{\partial G}{\partial T_i} \qquad E_m = \frac{\partial G}{\partial D_m} \tag{3g-17}$$

Since linear equations are obtained only if a permanent polarization P_0 is introduced, we assume that

$$D_3 = P_0 + D_3^* \tag{3g-18}$$

where D_3^* is a small variable component superposed on P_0. Also, D_1 and D_2 are small so that their squares and higher powers can be neglected compared with P_0. Introducing these into (3g-16) and differentiating, we have

$$\begin{aligned} S_1 &= s_{11}^DT_1 + s_{12}^D(T_2 + T_1) + Q_{12}(P_0^2 + 2P_0D_3^*) \\ S_2 &= s_{11}^DT_2 + s_{12}^D(T_1 + T_3) + Q_{12}(P_0^2 + 2P_0D_3^*) \\ S_3 &= s_{11}^DT_3 + s_{12}^D(T_1 + T_2) + Q_{11}(P_0^2 + 2P_0D_3^*) \\ S_4 &= 2(s_{11}^D - s_{12}^D)T_4 + 2(Q_{11} - Q_{12})P_0D_2 \\ S_5 &= 2(s_{11}^D - s_{12}^D)T_5 + 2(Q_{11} - Q_{12})P_0D_1 \\ S_6 &= 2(s_{11}^D - s_{12}^D)T_6 \\ E_1 &= -2(Q_{11} - Q_{12})P_0T_5 + D_1(\beta_{11}^T + 2K_{12}^TP_0^2 + 2K_{112}^TP_0^4) \\ E_2 &= -2(Q_{11} - Q_{12})P_0T_4 + D_2(\beta_{11}^T + 2K_{12}^TP_0^2 + 2K_{112}^TP_0^4) \\ E_3 &= -2Q_{11}P_0T_3 - 2Q_{12}P_0(T_1 + T_2) + D_3^*(\beta_{11}^T + 12K_{11}^TP_0^2 + 30K_{111}^TP_0^4) \end{aligned} \tag{3g-19}$$

It is obvious that the variable components of Eq. (3g-19) follow the same rule as for a piezoelectric crystal. There are three longitudinal modes and a shearing mode. The length longitudinal mode has the following constants:

$$\text{L.L. mode} \quad s_{11}^E = s_{11}^D\left[1 + \frac{4Q_{12}^2P_0^2}{\beta_{33}^T(P_0)s_{11}^D}\right] \qquad d_{31} = \frac{2Q_{12}P_0}{\beta_{33}^T(P_0)}$$

$$\epsilon_{33}^T(P_0) = \frac{1}{\beta_{33}(P_0)} \tag{3g-20}$$

where $\beta_{33}^T(P_0) = (\beta_{11}^T + 12K_{11}^TP_0^2 + 30K_{111}^TP_0^4)$ is the dielectric impermeability of the ceramic when it has a permanent polarization P_0

$$\text{L.T. bar} \quad s_{11}^E = s_{11}^D\left[1 + \frac{4Q_{11}^2P_0^2}{\beta_{33}^T(P_0)s_{11}^D}\right] \qquad d_{33} = \frac{2Q_{11}P_0}{\beta_{33}^T(P_0)}$$

$$\epsilon_{33}^T(P_0) = \frac{1}{\beta_{33}^T(P_0)} \tag{3g-21}$$

* If higher-order terms than those considered here are used, second-order electrostrictive and magnetostrictive terms and the change in elastic constants with polarization can be taken care of. For example, see W. P. Mason, *Phys. Rev.* **82** (5), 715–723 (June 1, 1951).

TABLE 3g-2. PROPERTIES OF CERAMICS AT 25°C

Material	d_{31} eff, coulombs/newton × 10^{11}	d_{33} eff, coulombs/newton × 10^{11}	$\epsilon_3^T(P_0)$ farads/m × 10^{11}	$Y_0^E = 1/s^E$ newtons/m² × 10^{11}	Shear stiffness $G^E = \frac{1}{2(s_{11}^E - s_{12}^E)}$ newtons/m² × 10^{11}	Numerics k_{31}	k_{33}	k_{15}	Energy stored $\frac{1}{2}(d_{33}^2/\epsilon)$ joules/m newton² × 10^{12}	Open-circuit voltage $g = d_{33}/\epsilon^T$ volts × m/newton	Force factor $d_{33}Y_0^E \times 10^{22}$ newton/volts × m
Commercial $BaTiO_3$ ceramic	−(5.6)	13–16	1,200–1,500	1.18	0.46	0.17	0.45	0.41	0.85	0.0106	13.5
97% $BaTiO_3$, 3% $CaTiO_3$	−(5.3)	13.5	1,230	1.22	0.47	0.17	0.43	0.39	0.75	0.0111	11.0
96% $BaTiO_3$, 4% $PbTiO_3$	−(3.8)	10.5	880	1.14	0.44	0.14	0.39	0.34	0.63	0.012	9.2
90% $BaTiO_3$, 4% $PbTiO_3$, 6% $CaTiO_3$	−(4.0)	11.5	710	1.24	0.48	0.167	0.48	0.43	0.93	0.016	9.3
84% $BaTiO_3$, 8% $PbTiO_3$, 8% $CaTiO_3$	−(2.7)	8.0	530	1.31	0.50	0.124	0.4	0.35	0.60	0.015	6.1
80% $BaTiO_3$, 12% $PbTiO_3$, 8% $CaTiO_3$	−(2.0)	6.0	400	1.28	0.49	0.113	0.34	0.3	0.45	0.015	4.7
	Data on Brush Ceramics*										
Ceramic A	−7.8	19.0	1,520	1.10	0.423	0.214	0.52	0.49	1.2	0.0126	20.9
5% $CaTiO_3$	−5.8	15.0	1,050	1.16	0.445	0.193	0.5	....	1.07	0.0142	17.4
5% $PbTiO_3$	−5.3	12.9	1,040	1.10	0.423	0.172	0.410	....	0.80	0.0124	14.2
9% $CaTiO_3$	−4.1	11.8	805	1.25	0.481	0.162	0.466	....	0.865	0.0147	14.8

* From H. Jaffe.

These formulas hold for a bar which is long in the direction of vibration compared with the cross-sectional dimensions. When a plate is used which is a number of wavelengths across, the sidewise motions S_1 and S_2 are zero and the constants are

$$\text{L.T. plate} \qquad \frac{1}{c_{11}^E} \qquad d'_{33} \qquad \epsilon'_{33}(P_0) \qquad \text{(3g-22)}$$

where

$$\frac{1}{c_{11}^E} = \frac{1}{c_{11}^P} + d'^2_{33}\epsilon'_{33}(P_0) \qquad d'_{33} = 2P_0\left(Q_{11} - \frac{2s_{12}^D}{s_{11}^D + s_{12}^D}Q_{12}\right)\epsilon'_{33}(P_0)$$

$$\epsilon^{T'}_{33}(P_0) = \frac{1}{\beta_{33}^T(P_0) + [4Q_{12}^2P_0^2/(s_{11}^D + s_{12}^D)]} \qquad \text{and} \qquad c_{11}^D = \frac{s_{11}^D + s_{12}^D}{(s_{11}^D - s_{12}^D)(s_{11}^D + 2s_{12}^D)}$$

The thickness shear mode has the fundamental constants $2(s_{11}^E - s_{12}^E)$; d_{14}; $\epsilon_{11}^T(P_0)$,

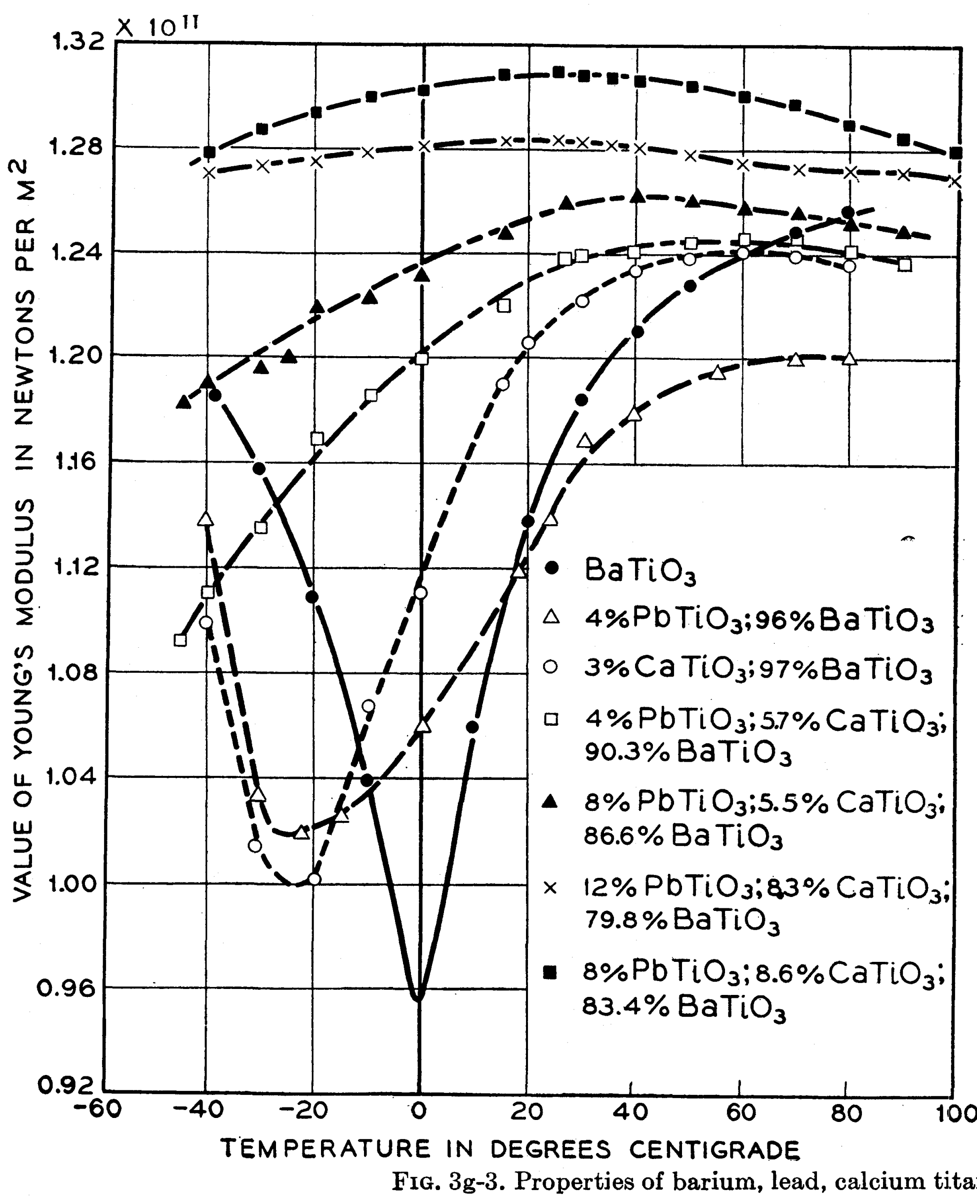
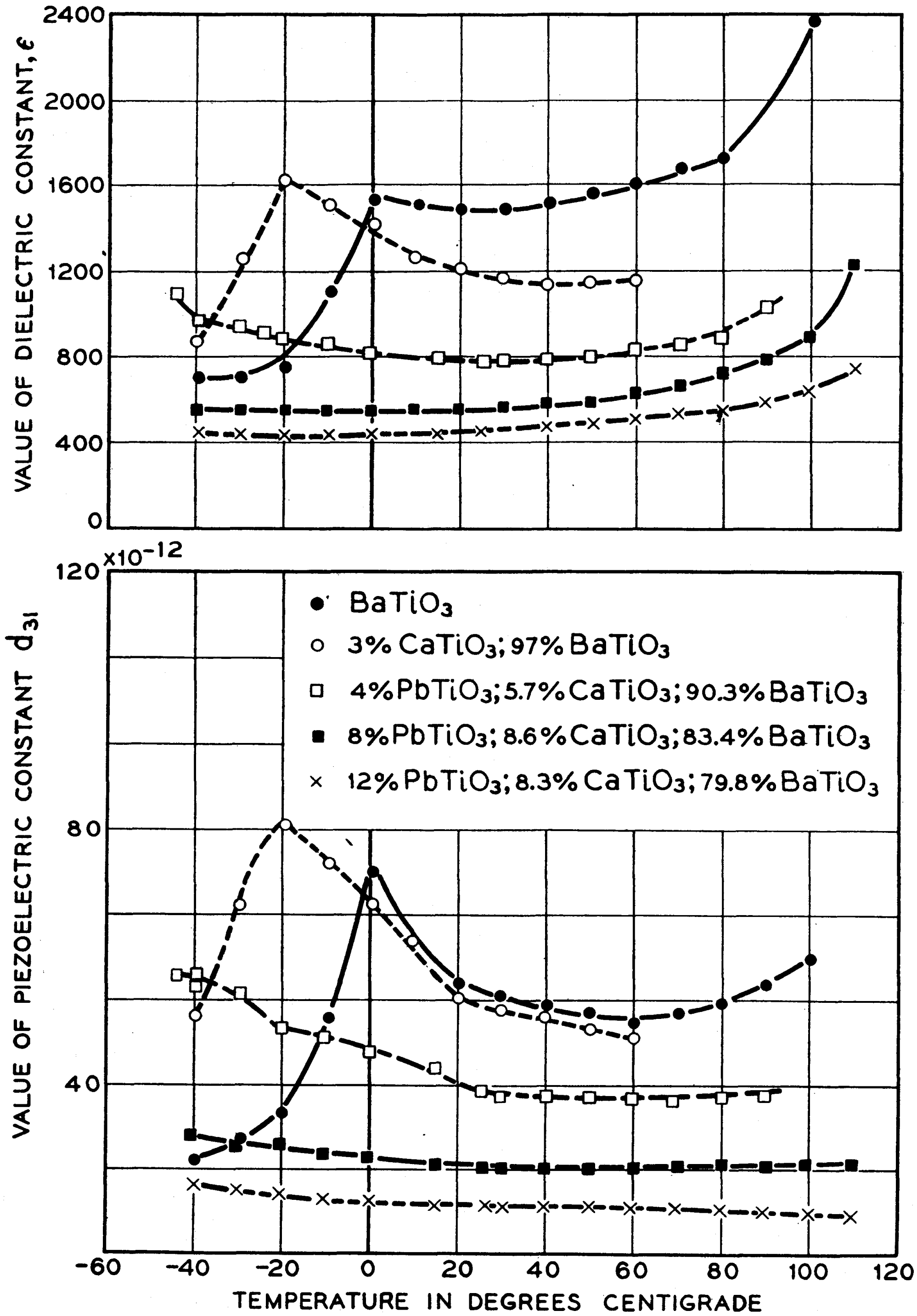


FIG. 3g-3. Properties of barium, lead, calcium titanate i.e., the dielectric constant perpendicular to the poling direction, where

$$d_{14} = \frac{2(Q_{11} - Q_{12})P_0}{\epsilon_{11}{}^T(P_0)} \qquad 2(s_{11}{}^E - s_{12}{}^E) = 2(s_{11}{}^D - s_{12}{}^D) + \frac{4(Q_{11} - Q_{12})^2 P_0{}^2}{\beta_{11}{}^T(P_0)}$$

$$\epsilon_{11}{}^T(P_0) = \frac{1}{(\beta_{11}{}^T + 2K_{12}{}^T P_0{}^2 + 2K_{112}{}^T P_0{}^4)} \qquad \text{(3g-23)}$$

Two other modes have been used in electrostrictive and magnetostrictive materials,



ceramics as functions of temperature and composition.

the radial mode and the torsional mode. The first is driven by polarizing the disk perpendicular to the major surface and involves the same fundamental constants as the length longitudinal mode of Eq. (3g-20). It has been shown[1] that the effective coupling and the resonant frequency of such disks are given by the equations

$$k^2 = \frac{2}{1-\sigma}\left(\frac{4Q_{11}{}^2P_0{}^2\epsilon^T(P_0)}{s_{11}{}^E}\right) \qquad f_R = \frac{2.03}{2\pi a}\sqrt{\frac{1}{s_{11}{}^E\rho(1-\sigma^2)}} \tag{3g-24}$$

where σ is Poisson's ratio which is approximately 0.3 for barium titanate ceramics. The torsional mode is generated in electrostrictive and magnetostrictive materials when the alternating displacement is at right angles to the polarization. This is easily accomplished for a magnetostrictive material by polarizing a cylinder radially by one set of windings and driving the cylinder by a set of windings coaxial with the cylinder. In an electrostrictive material, a torsional vibration can be obtained by inducing a permanent polarization in different directions on two sides of the cylinder and driving the cylinder by a set of two electrodes with the two gaps between them coming in the region of greatest permanent polarization. The fundamental elastic constant is the shear constant ($s_{44}{}^E = s_{55}{}^E$) while the fundamental piezoelectric constant is the shear piezoelectric constant d_{15} or the similar magnetostrictive constants.

Table 3g-2 gives some typical constants for a number of barium titanate compositions with lead and calcium titanate additions. Figure 3g-3 shows how the fundamental constants vary with temperature over a wide temperature range. Table 3g-3 gives some typical constants for a number of magnetostrictive materials.

3g-3. Equivalent Circuits for Magnetostrictive Transducers. The energy equation (3g-16) is the same for magnetostrictive and electrostrictive materials, provided the electric field and displacement are replaced by the magnetic field H and the magnetic flux density B. Hence the equivalent circuit of Fig. 3g-1 also applies to a magnetostrictive material, provided we replace E and i by $\int_0^l H_i\,dl = U$, the magnetomotive force and $\dot{B}S = \dot{\Phi}$ where S is the cross-sectional area, Φ the total flux through the magnetostrictive transducer, and $\dot{\Phi}$ the time rate of change of this flux. Hence all the fundamental quantities and coupling factors can be expressed in terms of the analogous quantities as shown by Table 3g-3. These hold for materials having a closed magnetic circuit such as a ring or a rod with closing magnetic circuit having a reluctance small compared with that for the rod. If this is not true, demagnetizing factors and additional reluctance values have to be taken account of and the value of Φ is the average value determined by all these factors.

In a transducer, however, it is not U and $\dot{\Phi}$ that we deal with, but rather the input voltage and current. These quantities are related by equations of the type

$$E = N\frac{d\Phi}{dt} \qquad U = Ni \tag{3g-25}$$

where N is the number of turns and the voltage, current, flux, and magnetomotive forces are directed as shown by Fig. 3g-4. These are the equations of a gyrator, shown by the symbol of Fig. 3g-4, which does not satisfy the reciprocity relationship. If we call Z_M the magnetic impedance defined by

$$Z_M = \frac{U}{d\Phi/dt} \tag{3g-26}$$

it is evident that the electrical impedance at the terminals of the transducer is equal to

$$Z_E = \frac{E}{i} = \frac{N^2}{Z_M} \tag{3g-27}$$

[1] W. P. Mason, "Piezoelectric Crystals and Their Application to Ultrasonics," chap. XII, D. Van Nostrand Company, Inc., New York, 1950.

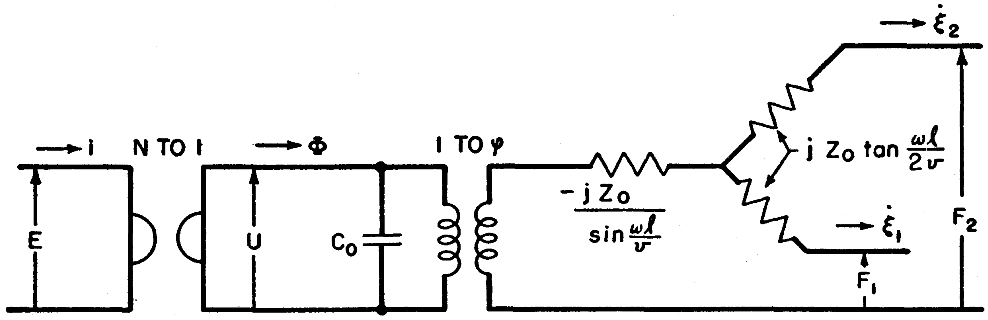
TABLE 3g-3. MAGNETOSTRICTIVE PROPERTIES OF METALS AND FERRITES
Data from C. M. Van der Burgt, *Phillips Research Repts.* **8**, 91–132, 1953

Material	$d_{33} \times 10^9$ webers/newton	$d_{14} \times 10^9$ webers/newton	Rev. per. long. $\mu^T(P_0) \times 10^4$ henrys/m	$Y_0^H = \frac{1}{s^H} \times 10^{-11}$ newtons/m²	k_{33}	Rev. per shear $\mu^T(P_0) \times 10^4$ henrys/m	Shear stiffness $G^E \times 10^{-11}$ newton/m²	Torsional coupling k_T	Energy stored $\frac{1}{2}(d_{33}^2/\mu^T) \times 10^{12}$ joules m/newton²	Density, kg/m³ $\times 10^{-3}$
99.9 nickel	−5.3	......	2.84	2.0	0.14	......	....	.....	0.05	8.9
50 Co; 0.5 Cr; 49.5 Fe	12.3	......	8.3	2.2	0.20	......	....	.....	0.09	8.2
35 Co; 0.5 Cr; 64.5 Fe	13.4	......	19.2	2.1	0.14	......	....	.....	0.047	8.1
NiO (15%); ZnO (35%); Fe_2O_3 (50%)	−11.1	−28.5	190	1.8	0.034	139	0.68	0.063	0.003	5.06
NiO (18%); ZnO (32%); Fe_2O_3 (50%)	−16.0	−39.5	77.5	1.62	0.073	74	0.62	0.115	0.0165	4.9
NiO (25%); ZnO (25%); Fe_2O_3 (50%)	−9.8	−20.3	22.0	1.53	0.082	20	0.59	0.110	0.022	4.85
NiO (32%); ZnO (18%); Fe_2O_3 (50%)	−8.7	−15.8	13.4	1.5	0.093	13.2	0.58	0.105	0.0282	4.85
NiO (40%); ZnO (10%); Fe_2O_3 (50%)	−5.9	−13.0	5.5	1.37	0.112	5.35	0.54	0.13	0.0315	4.76
NiO (50%); Fe_2O_3 (50%)	−4.4	......	2.8	0.93	0.08	2.4	0.36	0.09	0.0344	4.20

Data from R. M. Bozorth, E. A. Nesbit, and H. J. Williams

Material	Flux density B, webers/m²	Long. rev. per $\mu^T(P_0) \times 10^4$ henrys/m	Young's modulus $Y_0^A \times 10^{-11}$ newtons/m	Longitudinal coupling k_{33}	$d_{33} \times 10^9$ webers/newton	Energy stored $\frac{1}{2}(d_{33}^2/\mu^T) \times 10^{12}$ joules m/newton²	Density, kg/$m^3 \times 10^{-3}$
99.9% nickel	0.4	0.98	2.1	0.232	−5.0	0.127	8.9
	0.5	0.515	...	0.208	−3.26	0.103	
	0.55	0.317	...	0.177	−2.18	0.075	
45% Ni, 55% Fe, i.e., 45% Permalloy	0.722	8.94	1.6	0.154	11.5	0.074	8.17
	0.965	7.36	...	0.179	12.2	0.101	
	1.2	4.45	...	0.178	9.4	0.099	
	1.4	1.97	...	0.15	5.3	0.071	
2V Permindur, 2% V, 50% Co, 48% Fe	1.5	3.54	2.3	0.238	9.35	0.123	8.3
	1.6	2.61	...	0.222	7.5	0.108	
	1.8	2.23	...	0.202	6.3	0.089	
	2.0	1.14	...	0.18	4.0	0.07	

Hence the effect of the gyrator coupling is to invert all the elements of the equivalent circuit. Hence one should determine the element values of Fig. 3g-4 for the appropriate terminating conditions and then invert the values in accordance with Eq. (3g-27) to determine the elements of a magnetostrictive transducer. The values



$$C_0 = \frac{\mu^S l}{S} \quad ; \quad Z_0 = S\sqrt{\rho Y_0^H} \quad ; \quad v = \sqrt{\frac{Y_0^H}{\rho}} \quad ; \quad \varphi = \frac{d_{33} Y_0^H S}{l}$$

FIG. 3g-4. Equivalent circuit of a magnetostrictive rod.

given in Fig. 3g-4 are for a longitudinally vibrating rod where S is the cross-sectional area and l the length. μ^S is the average value of the permeability in the equations for the reluctance R

$$R = \frac{l}{\mu^S S} \tag{3g-28}$$

where μ^S is for the constant stress condition.

3h. Frequencies of Simple Vibrators. Musical Scales

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3h-1. Strings. The fundamental frequency of vibration of an ideal string is

$$f_0 = \frac{1}{2l}\sqrt{\frac{F}{m}} \tag{3h-1}$$

where f_0 is the frequency, l is the free length, F is the force (tension) stretching the string, and m is the mass per unit length. Values of m for steel and gut strings are given in Table 3h-1.

In addition to the vibration in a single loop which gives rise to the fundamental frequency, the ideal string may vibrate in harmonics whose frequencies are

$$f_n = nf_0 \tag{3h-2}$$

TABLE 3h-1. MASS PER UNIT LENGTH OF STEEL AND GUT STRINGS*

Diam		Steel, g/m	Gut, g/m	Diam		Steel, g/m	Gut, g/m	Diam		Steel, g/m	Gut, g/m
mm	in.			mm	in.			mm	in.		
0.20	0.0079	0.25	0.04	1.00	0.0394	6.15	1.10	1.80	0.0709	19.9	3.56
0.22	0.0087	0.30	0.05	1.02	0.0402	6.40	1.14	1.82	0.0717	20.4	3.64
0.24	0.0094	0.35	0.06	1.04	0.0409	6.65	1.19	1.84	0.0724	20.8	3.72
0.26	0.0102	0.42	0.07	1.06	0.0417	6.91	1.24	1.86	0.0732	21.3	3.80
0.28	0.0110	0.48	0.09	1.08	0.0425	7.17	1.28	1.88	0.0740	21.7	3.88
0.30	0.0118	0.55	0.10	1.10	0.0433	7.44	1.33	1.90	0.0748	22.2	3.97
0.32	0.0126	0.63	0.11	1.12	0.0441	7.71	1.38	1.92	0.0756	22.7	4.05
0.34	0.0134	0.71	0.13	1.14	0.0449	7.99	1.43	1.94	0.0764	23.1	4.14
0.36	0.0142	0.80	0.14	1.16	0.0457	8.27	1.48	1.96	0.0772	23.6	4.22
0.38	0.0150	0.89	0.16	1.18	0.0465	8.56	1.53	1.98	0.0780	24.1	4.31
0.40	0.0157	0.98	0.18	1.20	0.0472	8.86	1.58	2.00	0.0787	24.6	4.40
0.42	0.0165	1.08	0.19	1.22	0.0480	9.15	1.64	2.02	0.0795	25.1	4.49
0.44	0.0173	1.19	0.21	1.24	0.0488	9.46	1.69	2.04	0.0803	25.6	4.58
0.46	0.0181	1.30	0.23	1.26	0.0496	9.76	1.75	2.06	0.0811	26.1	4.67
0.48	0.0189	1.42	0.25	1.28	0.0504	10.1	1.80	2.08	0.0819	26.6	4.76
0.50	0.0197	1.54	0.27	1.30	0.0512	10.4	1.86	2.10	0.0827	27.1	4.85
0.52	0.0205	1.66	0.30	1.32	0.0520	10.7	1.92	2.12	0.0835	27.6	4.94
0.54	0.0213	1.79	0.32	1.34	0.0528	11.1	1.97	2.14	0.0843	28.2	5.04
0.56	0.0220	1.93	0.34	1.36	0.0535	11.4	2.03	2.16	0.0850	28.7	5.13
0.58	0.0228	2.07	0.37	1.38	0.0543	11.7	2.09	2.18	0.0858	29.2	5.23
0.60	0.0236	2.21	0.40	1.40	0.0551	12.1	2.16	2.20	0.0866	29.8	5.32
0.62	0.0244	2.36	0.42	1.42	0.0559	12.4	2.22	2.22	0.0874	30.3	5.42
0.64	0.0252	2.52	0.45	1.44	0.0567	12.8	2.28	2.24	0.0882	30.9	5.52
0.66	0.0260	2.68	0.48	1.46	0.0575	13.1	2.34	2.26	0.0890	31.4	5.62
0.68	0.0268	2.84	0.51	1.48	0.0583	13.5	2.41	2.28	0.0898	32.0	5.72
0.70	0.0276	3.01	0.54	1.50	0.0591	13.8	2.47	2.30	0.0906	32.5	5.82
0.72	0.0283	3.19	0.57	1.52	0.0598	14.2	2.54	2.32	0.0913	33.1	5.92
0.74	0.0291	3.37	0.60	1.54	0.0606	14.6	2.61	2.34	0.0921	33.7	6.02
0.76	0.0299	3.55	0.64	1.56	0.0614	15.0	2.68	2.36	0.0929	34.3	6.12
0.78	0.0307	3.74	0.67	1.58	0.0622	15.4	2.74	2.38	0.0937	34.8	6.23
0.80	0.0315	3.94	0.70	1.60	0.0630	15.7	2.81	2.40	0.0945	35.4	6.33
0.82	0.0323	4.14	0.74	1.62	0.0638	16.1	2.89	2.42	0.0953	36.0	6.44
0.84	0.0331	4.34	0.78	1.64	0.0646	16.5	2.96	2.44	0.0961	36.6	6.55
0.86	0.0339	4.55	0.81	1.66	0.0654	16.9	3.03	2.46	0.0968	37.2	6.65
0.88	0.0346	4.76	0.85	1.68	0.0661	17.4	3.10	2.48	0.0976	37.8	6.76
0.90	0.0354	4.98	0.89	1.70	0.0669	17.8	3.18	2.50	0.0984	38.4	6.87
0.92	0.0362	5.20	0.93	1.72	0.0677	18.2	3.25	2.52	0.0992	39.1	6.98
0.94	0.0370	5.43	0.97	1.74	0.0685	18.6	3.33	2.54	0.1000	39.7	7.09
0.96	0.0378	5.67	1.01	1.76	0.0693	19.0	3.41	2.56	0.1008	40.3	7.21
0.98	0.0386	5.91	1.06	1.78	0.0701	19.5	3.48	2.58	0.1016	40.9	7.32

* This table is based on a density of steel of 7.83 g/cm³. Density of gut is assumed to be 1.4 g/cm³, about one-sixth that of steel. This is only approximate, since the density of gut varies from sample to sample, and increases markedly with humidity. Brass wire has a density of 8.7 g/cm³, about 1.1 times that of steel.

where n is the integer denoting the particular mode of vibration. The length of each vibration loop is l/n. These successive lengths and the corresponding periods of vibration (i.e., the reciprocals of the frequencies) constitute a harmonic series according to the strict mathematical definition; nowadays, however, the frequencies themselves are usually said to make up a harmonic series.

The frequencies of actual strings depart somewhat from the frequencies computed from the simple formula because actual strings are stiff, they may be partially clamped at the ends, they are not infinitely thin, the tension increases with amplitude of vibration, the mass per unit length is not exactly uniform, there is internal damping and damping due to the surrounding air and supports, and the supports are not infinitely rigid. In the formulas which follow damping has been neglected.

For an actual string set

$$f = nf_0(1 + G) \tag{3h-3}$$

where the factor $(1 + G)$ is a measure of the departure (i.e., the inharmonicity) from the ideal harmonic values. Table 3h-2 lists values of G for various small perturbations. The approximations are valid only when G is small.

TABLE 3h-2. PERTURBATION IN FREQUENCY OF A STRING

Cause	G	Explanation
Stiffness	$\frac{n^2\pi^3 d^4 Y}{128 l^2 F}$	Y is Young's modulus, d is the diameter of the string
Yielding support	$\frac{4ml}{4\pi^2 n^2 M - K/f_0^2}$	The support consists of a mass M on a spring of transverse force constant K. Multiply by 2 if there are two such supports
Variable density	$-\frac{1}{l}\int_0^l g(x)\sin^2\frac{(\pi n x)}{l}\,dx$	The mass per unit length is $m = m_0[1 + g(x)]$ where m_0 is the mean value over the string and x is the distance from one end of the string; the function $g(x)$ must be small in comparison with unity

For musical purposes it is often convenient to give the inharmonicity in cents (hundredths of an equally tempered semitone) by setting

$$1 + G = 2^{\delta/1{,}200} = e^{\delta/1{,}731} \tag{3h-4}$$

where δ is the inharmonicity. To a usually acceptable approximation, $\delta = 1{,}731G$.

If the stiff string listed in Table 3h-2 is of steel music wire, $Y/\rho = 25.5 \times 10^6$ m^2/sec^2, Y being Young's modulus and ρ the density. The tension is very nearly $F = l^2\rho f_0^2\pi d^2$. Thus for steel wire, and by virtue of the stiffness formula, the inharmonicity in cents is $\delta = 3.4 \times 10^{13} d^2/f_0^2 l^4$, provided that the diameter and length are in centimeters.

3h-2. Air Columns and Rods. The air within a simple tube of constant cross section, open at both ends or closed at both ends, vibrates freely at a frequency near

$$f = \frac{nc}{2l} \tag{3h-5}$$

where n is an integer (mode of vibration number), c is the speed of sound in the contained air, and l is the length of the tube (see Sec. 3d for speed of sound in air and its dependence on temperature). The diameter of the tube must be relatively small;

plane sound waves propagated longitudinally are assumed. The same formula applies to thin rods vibrating longitudinally and suitably supported (say, at distances $l/2n$ from the ends) so that the vibration is not inhibited (see Sec. 3f for speed of sound in solids).

Open organ pipe is an example of a doubly open tube of constant cross section. To calculate its frequency adequately it must be recognized, however, that the air beyond the physical ends of the tube partakes of the vibration and adds inertia to the vibrating system. (This does *not* mean, however, that there is a velocity antinode beyond the end of the tube.) The necessary corrections to the simple formula are usually introduced as empirical "end corrections" to be added to the geometrical length; thus

$$f = \frac{nc}{2(l + x_1 + x_2)} \tag{3h-6}$$

where $x_1 = 0.3d$ is the correction for the unimpeded end (d being the inside diameter of the pipe) and $x_2 = 1.4d$ is the correction for the mouth of the pipe. These are rough approximations; the literature on the end correction is extensive.[1]

The air inside a cylindrical tube that is closed at one end and open at the other vibrates at frequency

$$f = \frac{nc}{4(l + x)} \tag{3h-7}$$

where $x = x_1$ if the open end is unimpeded. In the case of the "closed" organ pipe (meaning closed at one end only), $x = x_2$.

The air in a conical tube is resonant in some cases at the same frequencies as a doubly open cylindrical tube of the same length, but there is the important difference that the contained sound waves are spherical rather than plane. Table 3h-3 gives equations[2] to be solved for each combination of end conditions; $k = 2\pi f/c$. "Closed-open," for example, means that the smaller end of the truncated cone is closed while the larger end is open; r_1 is the slant distance from the extrapolated apex of the cone to the smaller end and r_2 is the slant distance to the larger end. The slant length of the resonator is thus $r_2 - r_1$. When $r_1 = 0$, the length is r_2 and the cone is complete to the apex. Formulas for computing frequency when the cone is complete are shown at the right of Table 3h-3. As in the case of cylindrical tubes, the length should be

TABLE 3h-3. FREQUENCIES OF CONICAL RESONATORS

Ends	Equation	For $r_1 = 0$
Closed-closed	$kr_2 - \tan^{-1} kr_2 = kr_1 - \tan^{-1} kr_1$	$\tan kr_2 = kr_2$
Closed-open	$\tan k(r_2 - r_1) = -kr_1$	$f_1 = \frac{nc}{2r_2}$
Open-closed	$\tan k(r_2 - r_1) = kr_2$	$\tan kr_2 = kr_2$
Open-open	$f = \frac{nc}{2(r_2 - r_1)}$	$f = \frac{nc}{2r_2}$

slightly modified by end corrections. As the angle of the cone increases the correction decreases and may even become negative.[3]

3h-3. Volume Resonators. The Helmholtz resonator consists of a nearly closed cavity of volume V with an opening of acoustical conductance C. If the opening is

[1] E. G. Richardson, ed., "The Technical Aspects of Sound," vol. I, pp. 493–496, 578, Elsevier Publishing Company, Amsterdam, 1953; Harold Levine, *J. Acoust. Soc. Am.* **26,** 200–211 (1954).

[2] Eric J. Irons, *Phil. Mag.* **9,** 346–360 (1930).

[3] A. E. Bate and E. T. Wilson, *Phil. Mag.* **26,** 752–757 (1938).

in a thin wall the conductance is simply d, the diameter of the hole. If the opening is through a short neck of length l, approximately

$$C = \frac{\pi d^2}{4(l + 0.8d)} \tag{3h-8}$$

The natural frequency of the resonantor is

$$f = \frac{c}{2\pi}\sqrt{\frac{C}{V}} \tag{3h-9}$$

the velocity of sound in the opening being c. The equation is valid for wavelengths large in comparison with the dimensions of the resonator.

The ocarina may be recognized as an instrument of the resonator type because the *position* of an open hole of given size is immaterial; when the holes are all equal they may be opened in any order to give the same scale. The total conductance for use in the formula given above is the sum of the conductance of individual holes, provided that they are separated far enough that there is no interaction.

3h-4. Bars. A long thin bar clamped and/or free at the end(s) can vibrate transversely at the fundamental frequencies listed in Table 3h-4 under mode 1. The length of the bar is l, Y is Young's modulus, ρ is the density, and κ is the radius of gyration about the neutral axis of the cross section. For a round bar $\kappa = d/4$, where d is the diameter. For a flat bar of thickness t (in the plane of vibration) $\kappa = t/\sqrt{12}$; the width is immaterial. The frequency of a bar clamped at both ends is the same as that of a bar free at both ends. The frequency of a higher mode of vibration can be found by multiplying the fundamental frequency by the ratio indicated in Table 3h-4;

TABLE 3h-4. FREQUENCIES OF TRANSVERSE VIBRATION OF BARS

Ends	Frequency	Ratio			Cents		
	Mode → 1	2	3	4	2	3	4
Clamped-free	$f_1 = \frac{0.5597\kappa}{l^2}\sqrt{\frac{Y}{\rho}}$	6.267	17.548	34.387	3,177	4,960	6,124
Free-free, or clamped-clamped	$f_1 = \frac{3.561\kappa}{l^2}\sqrt{\frac{Y}{\rho}}$	2.756	5.404	8.933	1,755	2,921	3,791

the intervals in cents corresponding to these ratios are given at the extreme right of the table. These are the classic[1] values for thin bars; the frequencies of actual bars are lowered slightly as a consequence of rotatory inertia, lateral inertia, and shear.[2] For example, for a steel bar whose length is 40 times the thickness, the frequencies of the first four modes of vibration are expected to be 0.997, 0.992, 0.984, and 0.974 times the corresponding "thin" values (i.e., lowered 5, 14, 28, and 46 cents, respectively).

The simple tuning fork may be recognized as an example of dual clamped-free bars. The frequency of a tuning fork made of ordinary steel may be computed approximately from

$$f = \frac{80{,}000t}{l^2} \tag{3h-10}$$

[1] Lord Rayleigh, "Theory of Sound," vol. I, p. 280, Macmillan & Co., Ltd., London, 1894. The interval erroneously given as 2.4359 octaves has been corrected here to 2.4340 octaves = 2,921 cents.

[2] William T. Thomson, *J. Acoust. Soc. Am.* **11**, 199–204 (1939). There is an error: $m = \beta/[1 + \beta^2(k/L)^2]^{\frac{1}{4}}$, not $m = \beta/[1 + \beta^2(k/L)^2]^{\frac{1}{2}}$.

TABLE 3h-5. FREQUENCIES OF THE EQUALLY TEMPERED SCALE, BASED ON THE INTERNATIONAL STANDARD A = 440 CPS

Note	S	f	$2\pi f$	Note	S	f	$2\pi f$	Note	S	f	$2\pi f$
C_0	0	16.352	102.74	C_3	36	130.81	821.92	C_6	72	1,046.5	6,575.4
	1	17.324	102.74		37	138.59	870.79		73	1,108.7	6,966.4
D_0	2	18.354	115.32	D_3	38	146.83	922.58	D_6	74	1,174.7	7,380.6
	3	19.445	122.18		39	155.56	977.43		75	1,244.5	7,819.5
E_0	4	20.602	129.44	E_3	40	164.81	1,035.6	E_6	76	1,318.5	8,284.4
F_0	5	21.827	137.14	F_3	41	174.61	1,097.1	F_6	77	1,396.9	8,777.1
	6	23.125	145.30		42	185.00	1,162.4		78	1,480.0	9,299.0
G_0	7	24.500	153.93	G_3	43	196.00	1,231.5	G_6	79	1,568.0	9,851.9
	8	25.957	163.09		44	207.65	1,304.7		80	1,661.2	10,438
A_0	9	27.500	172.59	A_3	45	220.00	1,382.3	A_6	81	1,760.0	11,058
	10	29.135	183.06		46	233.08	1,464.5		82	1,864.7	11,716
B_0	11	30.868	193.95	B_3	47	246.94	1,551.6	B_6	83	1,975.5	12,413
C_1	12	32.703	205.48	C_4	48	261.63	1,643.8	C_7	84	2,093.0	13,151
	13	34.648	217.70		49	277.18	1,741.6		85	2,217.5	13,933
D_1	14	36.708	230.64	D_4	50	293.66	1,845.2	D_7	86	2,349.3	14,761
	15	38.891	244.36		51	311.13	1,954.9		87	2,489.0	15,639
E_1	16	41.203	258.89	E_4	52	329.63	2,071.1	E_7	88	2,637.0	16,569
F_1	17	43.654	274.28	F_4	53	349.23	2,194.3	F_7	89	2,793.8	17,554
	18	46.249	290.59		54	369.99	2,324.7		90	2,960.0	18,598
G_1	19	48.999	307.87	G_4	55	392.00	2,463.0	G_7	91	3,136.0	19,704
	20	51.913	326.18		56	415.30	2,609.4		92	3,322.4	20.875
A_1	21	55.000	345.58	A_4	57	440.00	2,764.6	A_7	93	3,520.0	22,117
	22	58.270	366.12		58	466.16	2,929.0		94	3,729.3	23,432
B_1	23	61.735	387.90	B_4	59	493.88	3,103.2	B_7	95	3,951.1	24,825
C_2	24	65.406	410.96	C_5	60	523.25	3,287.7	C_8	96	4,186.0	26,301
	25	69.296	435.40		61	554.37	3,483.2		97	4,434.9	27,865
D_2	26	73.416	461.29	D_5	62	587.33	3,690.3	D_8	98	4,698.6	29,522
	27	77.782	488.72		63	622.25	3,909.7		99	4,978.0	31,278
E_2	28	82.407	517.78	E_5	64	659.26	4,142.2	E_8	100	5,274.0	33,138
F_2	29	87.307	548.57	F_5	65	698.46	4,388.5	F_8	101	5,587.7	35,108
	30	92.499	581.19		66	739.99	4,649.5		102	5,919.9	37,196
G_2	31	97.999	615.74	G_5	67	783.99	4,926.0	G_8	103	6,271.9	39,408
	32	103.83	652.36		68	830.61	5,218.9		104	6,644.9	41,751
A_2	33	110.00	691.15	A_5	69	880.00	5,529.2	A_8	105	7,040.0	44,234
	34	116.54	732.25		70	932.33	5,858.0		106	7,458.6	46,864
B_2	35	123.47	775.79	B_5	71	987.77	6,206.3	B_8	107	7,902.1	49,651

Numerous subscript notations have been employed to distinguish the notes of one octave from those of another. The particular scheme used here assigns to C_0 a frequency which corresponds roughly to the lowest pitch. S is the number of semitones counted from this C_0.

provided that the thickness t and length l of the prongs are given in centimeters.

It is evident from Table 3h-4 that the different modes of vibration of a uniform bar are inharmonic. However, the cross section of the bar in the modern xylophone or marimba is often given an empirical lengthwise "undulation" such that the second mode of vibration of the free-free bar is changed in frequency to 3 or 4 times the fundamental frequency.[1] The frequencies of the higher modes of vibration are also modified

TABLE 3h-6. INTERVALS IN CENTS CORRESPONDING TO CERTAIN FREQUENCY RATIOS

Name of interval	Frequency ratio	Cents
Unison	1:1	0
Minor second or semitone	1.059463:1	100
Semitone	16:15	111.731
Minor tone or lesser whole tone	10:9	182.404
Major second or whole tone	1.122462:1	200
Major tone or greater whole tone	9:8	203.910
Minor third	1.189207:1	300
Minor third	6:5	315.641
Major third	5:4	386.314
Major third	1.259921:1	400
Perfect fourth	4:3	498.045
Perfect fourth	1.334840:1	500
Augmented fourth	45:32	590.224
Augmented fourth	1.414214:1	600
Diminished fifth	1.414214:1	600
Diminished fifth	64:45	609.777
Perfect fifth	1.498307:1	700
Perfect fifth	3:2	701.955
Minor sixth	1.587401:1	800
Minor sixth	8:5	813.687
Major sixth	5:3	884.359
Major sixth	1.681793:1	900
Harmonic minor seventh	7:4	968.826
Grave minor seventh	16:9	996.091
Minor seventh	1.781797:1	1,000
Minor seventh	9:5	1,017.597
Major seventh	15:8	1,088.269
Major seventh	1.887749:1	1,100
Octave	2:1	1,200.000

by variation in cross section for special purposes such as the simulation of the sound of a bell.[2]

3h-5. Musical Scales. By international agreement the standard tuning frequency for musical performance is the A of 440 cps. The frequencies of the equally tempered scale based on this frequency appear in Table 3h-5. Middle C thus has a frequency of 261.6 cps. The C of 256 cps, frequently used in the past for demonstrations in physics, has never been adopted for practical musical performance.

[1] See U.S. Pats. 1,838,502 (1931) and 1,632,751 (1927).

[2] See U.S. Pats. 2,273,333 (1942), 2,516,725 (1950), 2,536,800 (1951), and 2,606,474 (1952).

For many calculations with musical intervals it is convenient to deal with logarithmic units that can be added instead of the ratios which must be multiplied. The octave is equal to 1,200 logarithmic cents, and the equally tempered semitone is 100 cents. The interval in cents corresponding to any two frequencies f_1 and f_2 is $1{,}200 \log_2 (f_2/f_1) = 3{,}986 \log_{10} (f_2/f_1)$. Table 3h-6 lists certain common intervals in cents, and the corresponding ratios; the frequency ratios for intervals up to 100 cents are given in Table 3h-7.

TABLE 3h-7. RATIOS FOR INTERVALS TO 100 CENTS

Cents	Ratio	Cents	Ratio	Cents	Ratio	Cents	Ratio
0	1.000000	25	1.014545	50	1.029302	75	1.044274
1	1.000578	26	1.015132	51	1.029896	76	1.044877
2	1.001158	27	1.015718	52	1.030492	77	1.045481
3	1.001734	28	1.016305	53	1.031087	78	1.046085
4	1.002313	29	1.016892	54	1.031683	79	1.046689
5	1.002892	30	1.017480	55	1.032079	80	1.047294
6	1.003472	31	1.018068	56	1.032876	81	1.047899
7	1.004052	32	1.018656	57	1.033473	82	1.048505
8	1.004632	33	1.019244	58	1.034070	83	1.049111
9	1.005212	34	1.019833	59	1.034667	84	1.049717
10	1.005793	35	1.020423	60	1.035265	85	1.050323
11	1.006374	36	1.021012	61	1.035863	86	1.050930
12	1.006956	37	1.021602	62	1.036462	87	1.051537
13	1.007537	38	1.022192	63	1.037060	88	1.052145
14	1.008120	39	1.022783	64	1.037660	89	1.052753
15	1.008702	40	1.023374	65	1.038259	90	1.053361
16	1.009285	41	1.023965	66	1.038859	91	1.053970
17	1.009868	42	1.024557	67	1.039459	92	1.054579
18	1.010451	43	1.025149	68	1.040060	93	1.055188
19	1.011035	44	1.025741	69	1.040661	94	1.055798
20	1.011619	45	1.026334	70	1.041262	95	1.056408
21	1.012204	46	1.026927	71	1.041864	96	1.057018
22	1.012789	47	1.027520	72	1.042466	97	1.057629
23	1.013374	48	1.028114	73	1.043068	98	1.058240
24	1.013959	49	1.028708	74	1.043671	99	1.058851

3i. Radiation of Sound

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3i-1. Introduction. Radiation of sound may take place in a number of ways but, basically, all sound generators cause an alternating pressure to be set up in the fluid medium within which the sound energy is established. The sound energy that is set up in a medium depends not only on the physical characteristics of the medium and the oscillatory volume displacement of the fluid set up by the vibrating source but also upon the size and shape of the generator. The acoustic power generated by any vibrating source can be expressed by

$$P = U^2 R_A \times 10^{-7} \qquad \text{watts} \tag{3i-1}$$

where U = rate of volume displacement of the fluid, cc/sec
R_A = acoustic radiation resistance of the source, acoustic ohms

If the rate of volume displacement is taken in peak cc/sec, Eq. (3i-1) will yield peak watts of power. If the volume displacement is taken in rms cc/sec, the power will be given in rms watts.

Of the many possible methods for generating sound, two types of generators will effectively serve to classify most of them. These basic generators are (1) pulsating sphere, and (2) vibrating piston.

Each type of generator has a different acoustic impedance characteristic which depends on the dimensions of the source and on the frequency of vibration.

3i-2. Acoustic Impedance. *Pulsating Sphere.* The specific acoustic impedance of a pulsating sphere is given by

$$z = \frac{\rho c}{1 + [1/(\pi D/\lambda)]^2} + j\frac{\rho c/(\pi D/\lambda)}{1 + [1/(\pi D/\lambda)]^2} \qquad \text{acoustic ohms/cm}^2 \tag{3i-2}$$

where ρ = density of the medium, g/cc
c = velocity of sound in the medium, cm/sec
D = diameter of the sphere, cm
$\lambda = c/f$
f = frequency, cps

It can be seen from inspection that at high frequencies, where D/λ becomes very large, the specific acoustic impedance becomes a pure resistance equal to ρc and the reactance term vanishes. At low frequencies, where D/λ is small, the specific acoustic impedance becomes

$$z = \rho c\left(\frac{\pi D}{\lambda}\right)^2 + j\rho c\frac{\pi D}{\lambda} \qquad \text{acoustic ohms/cm}^2 \tag{3i-3}$$

A plot of the specific acoustic resistance and reactance of a pulsating sphere as a function of D/λ is shown in Fig. 3i-1. To obtain the total acoustic radiation resistance

R_A of the sphere, it is necessary to divide the specific acoustic resistance by the total surface area of the sphere in cm². The value of R_A thus determined, when substituted in Eq. (3i-1), will give the actual acoustic watts being generated by the spherical source.

Vibrating Piston. The specific acoustic impedance of a circular piston set in an infinite rigid baffle and radiating sound from one of its surfaces is given by

$$z = \rho c \left[1 - \frac{J_1(2\pi D/\lambda)}{\pi D/\lambda} \right] + j\rho c \frac{K_1(2\pi D/\lambda)}{2(\pi D/\lambda)^2} \quad \text{acoustic ohms/cm}^2 \qquad \text{(3i-4)}$$

where D is the diameter of the piston in centimeters, J_1 and K_1 are Bessel functions, and the remaining symbols are defined under Eq. (3i-2).

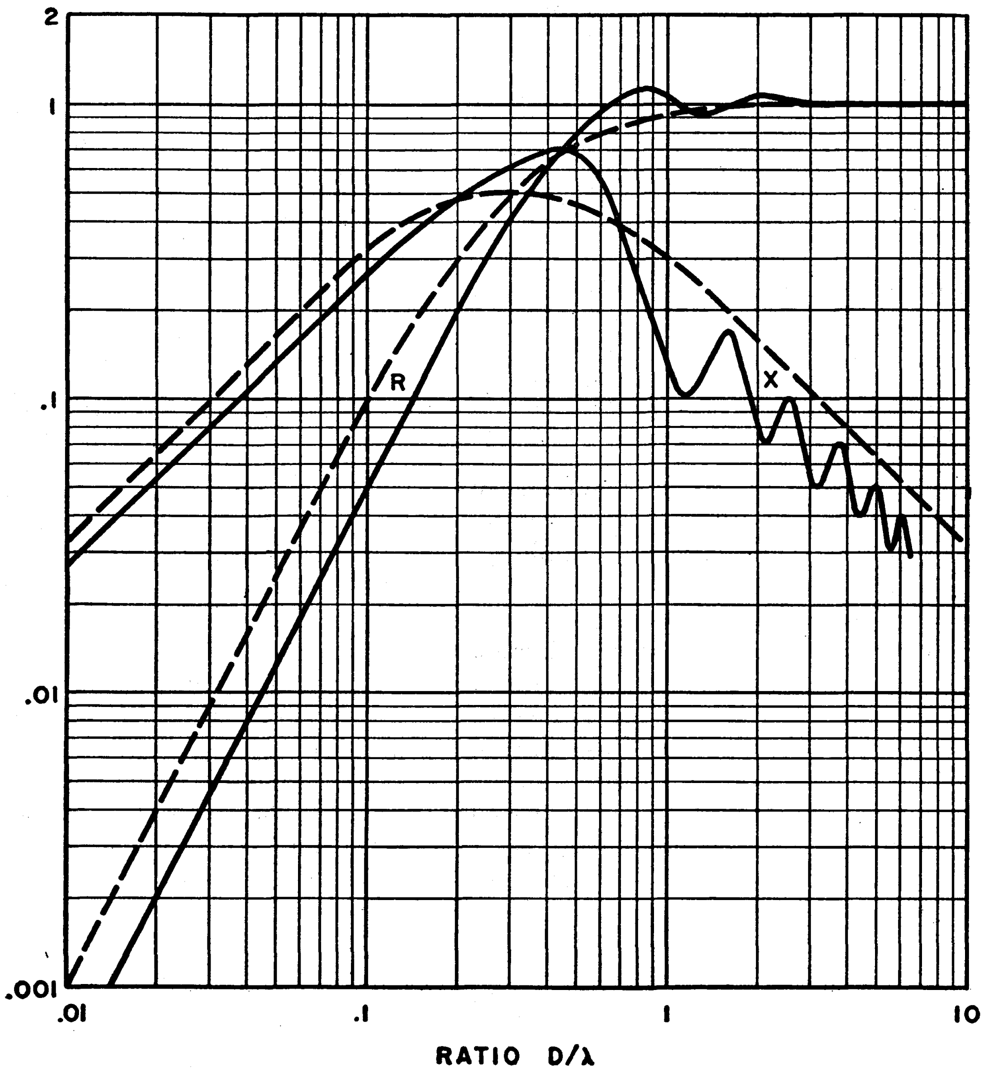

FIG. 3i-1. Specific acoustic resistance R and reactance X of a pulsating sphere (dashed curves) and a vibrating piston set in an infinite baffle (solid curves). To obtain magnitude of R or X multiply ordinates by ρc of the medium.

At high frequencies, where D/λ is large, Eq. (3i-4) reduces to a pure resistance equal to ρc. At low frequencies, where D/λ is small, the specific acoustic impedance for a piston set in an infinite baffle with one side radiating becomes

$$z = \frac{\rho c(\pi D/\lambda)^2}{2} + j\rho c \frac{8D}{3\lambda} \quad \text{acoustic ohms/cm}^2 \qquad \text{(3i-5)}$$

A plot of the specific acoustic resistance and reactance for a vibrating piston mounted in an infinite baffle is shown in Fig. 3i-1. To obtain the total acoustic radiation resistance of the piston, it is necessary to divide the specific resistance by the piston area in cm². The value of R_A so determined, when substituted in Eq. (3i-1), will give the actual acoustic watts being generated by a piston.

Summary of Radiation Impedance Characteristics. In Table 3i-1 are shown the magnitudes of the acoustic radiation resistance and reactance for a sphere and piston for both low-frequency (D/λ small) and high-frequency (D/λ large) operation.

TABLE 3i-1. TABULATED VALUES OF THE TOTAL ACOUSTIC RADIATION RESISTANCE AND REACTANCE OF A SPHERE AND PISTON IN ACOUSTIC OHMS

	$D/\lambda \ll 1$		$D/\lambda \gg 1$	
	R_A	X_A	R_A	X_A
Pulsating sphere	$\rho c \frac{\pi}{4\lambda^2}$	$\frac{\rho c}{\pi D\lambda}$	$\frac{\rho c}{A}$	0
Vibrating piston (in infinite baffle)	$\rho c \frac{\pi}{2\lambda^2}$	$\rho c \frac{8}{3\pi D\lambda}$	$\frac{\rho c}{A}$	0

ρ = density of the medium, g/cm²
c = velocity of sound in the medium, cm/sec
λ = wavelength of sound in the medium, cm
$\lambda = c/f$
f = frequency of the sound vibration, cps
D = diameter of sphere or pistion, cm
A = surface area of sphere or piston, cm²

3i-3. Directional Radiation of Sound. Whenever sound energy is generated from a source whose dimensions are small compared with the wavelength of the vibration in the medium, the intensity will be uniform in all angular directions and the generator is generally defined as a point source. When the dimensions of the vibrating surface are large compared with the wavelength, phase interferences will be experienced at different points in space due to the differences in time arrival of the vibrations originating from different portions of the surface, which results in a nonuniform directional radiation pattern. Practical use is made of this phenomena when it is desired to produce special directional patterns by arranging the geometry and size of the vibrating surfaces of a sound generator to create the desired characteristic.

In many instances, a transmitter is designed so that the sound is radiated in a relatively sharp beam so that the energy is concentrated only within a specific desired angular region. When such a directional structure is employed as a receiver, the transducer will be more capable of picking up weak signals from a specified direction than would be the case from a nondirectional transducer. The reason for this improvement is the reduced sensitivity of the directional receiver to random background noises that will be present in all directions from the source. The number of decibels by which the signal-to-noise ratio is improved by a directional receiver over a nondirectional receiver is known as the *directivity index* of the transducer. It will be defined more fully later. The following will show the directional radiation characteristics of several common structures.

Uniform Line Source. If a uniform long line is vibrating at uniform amplitude, the radiated sound intensity will be a maximum in a plane which is the perpendicular bisector of the line. At angles removed from the perpendicular bisector of the line, the intensity will fall off to a series of nulls and secondary maxima of diminishing amplitudes as the angle of incidence to the axis of the line deviates from the normal bisector of the line. For a line of length L vibrating uniformly over its entire length at a frequency corresponding to a wavelength of sound λ in the medium, the ratio of the sound pressure p_Θ produced at an angle Θ removed from the normal axis of maximum response to the sound pressure p_0 on the normal axis is given by

$$\frac{p_\Theta}{p_0} = \frac{\sin\left[(\pi L/\lambda)\sin\Theta\right]}{(\pi L/\lambda)\sin\Theta} \tag{3i-6}$$

If L is large compared with λ, the response as a function of Θ will go through a series of nulls and secondary maxima of successively diminishing amplitudes.

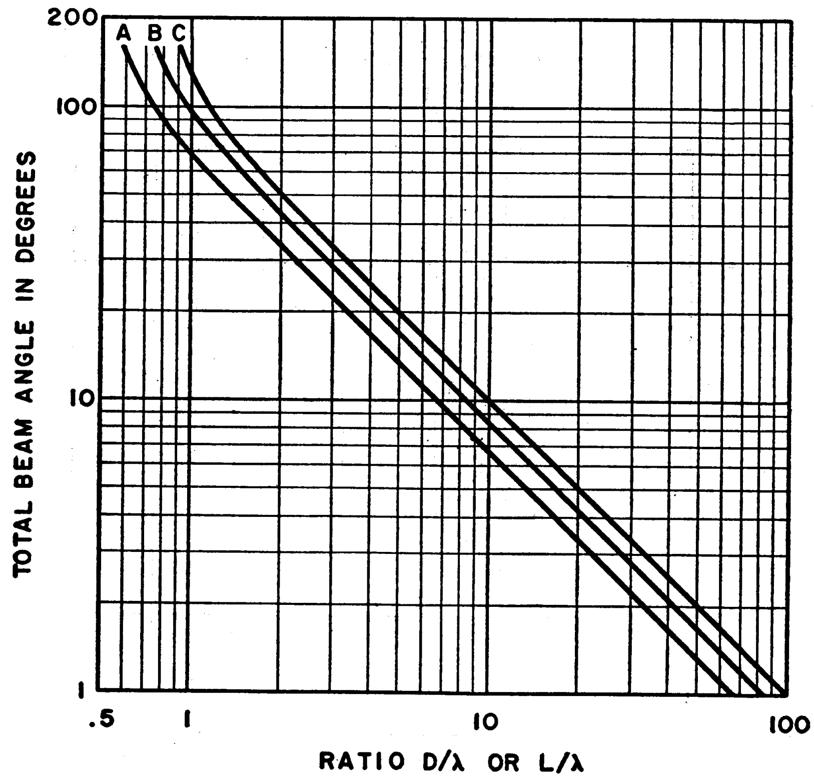

FIG. 3i-2. Total beam angle for a piston, ring, and line source as a function of size of source to wavelength of sound being radiated. *A*, thin ring of diameter *D*. *B*, uniform line of length *L*. *C*, piston of diameter *D*. (*Curves A and C from Massa, "Acoustic Design Charts," The Blakiston Division, McGraw-Hill Book Company, Inc., New York, 1942.*)

Circular Piston in Infinite Baffle. The directional radiation pattern from a large circular piston vibrating at constant amplitude and phase and set into an infinite rigid baffle may be obtained from the expression

$$\frac{p_\Theta}{p_0} = \frac{2J_1\left[(\pi D/\lambda)\sin\Theta\right]}{(\pi D/\lambda)\sin\Theta} \tag{3i-7}$$

where p_Θ = sound pressure at an angle Θ from the normal axis of the piston
p_0 = sound pressure on normal axis of piston
D = diameter of piston
λ = wavelength of sound
J_1 = Bessel function of order 1

From this equation, it can be seen that, as D/λ increases, the beam width becomes smaller and the sound pressure goes through a series of nulls and secondary maxima as Θ progressively departs from the normal axis to the piston.

Thin Circular Ring. The directional radiation pattern from a large narrow circular ring of diameter D vibrating at constant amplitude and fitted into an infinite plane baffle may be obtained from the expression

$$\frac{p_\Theta}{p_0} = J_0\left(\frac{\pi D}{\lambda}\sin\Theta\right) \tag{3i-8}$$

where J_0 = Bessel function of order zero and all other symbols are defined under Eq. (3i-7).

Beam Width for Line, Piston, and Ring. From Eqs. (3i-6), (3i-7), and (3i-8), the total beam width has been computed for the radiation from each of the three types of sound generators. The total beam width is here defined as the angle 2Θ at which the pressure p_Θ is reduced 10 db in magnitude from the maximum on axis response p_0. By setting p_Θ/p_0 equal to -10 db or 0.316 in magnitude in these equations, the three curves plotted in Fig. 3i-2 were computed.

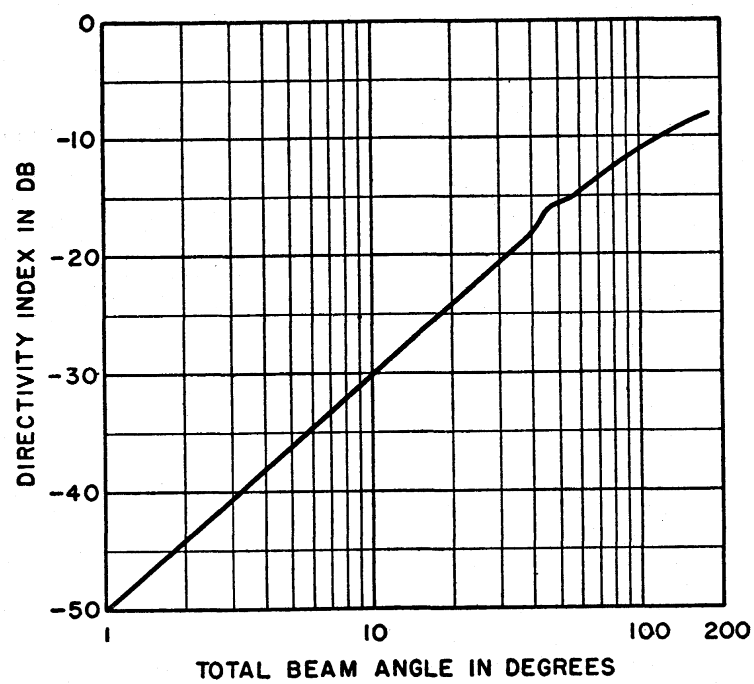

FIG. 3i-3. Directivity index of a piston or ring as a function of total beam angle where beam angle is defined as the included angle of the main beam between the 10-decibel-down points in the directional response. (*Computed from Massa, "Acoustic Design Charts," The Blakiston Division, McGraw-Hill Book Company, Inc., New York, 1942.*)

3i-4. Directivity Index. It has already been mentioned that a directional transducer has an advantage over a nondirectional structure whenever it is desired to send or receive signals from a particular localized direction only. The fact that the directional transducer is less sensitive to sounds coming from random undesired directions makes it possible for it to detect weaker signals than would be possible with a nondirectional unit. The measure of this improvement in decibels corresponds to the *directivity index* of the transducer. The directivity index of a transducer is defined as the ratio of the total power radiated by a transducer to the total power required by a nondirectional transducer to produce the same peak intensity as is produced by the directional transducer on its axis of maximum response.

The directivity index of a transducer is expressed in decibels, and a plot of the directivity index as a function of beam width for a piston or ring is shown in Fig. 3i-3.

3j. Architectural Acoustics

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3j-1. Sound-absorptive Materials. When sound waves strike a surface, the energy may be divided into three portions: the incident, reflected, and absorbed energy. Suppose plane waves are incident on a surface of infinite extent. For this case, the absorption coefficient α of the surface may be defined as

$$\alpha = \frac{\int_s \mathbf{I}_p \cdot \mathbf{ds}}{\int_s \mathbf{I}_A \cdot \mathbf{ds}} \tag{3j-1}$$

where I_p is the time average of the intensity vector of the sound field at the absorptive surface; ds is the vector surface element—the positive direction being into the material from the incident side; and I_A is the time average of the intensity vector which would exist at the surface element if the surface were removed. The absorption coefficient defined above is a function of angle of incidence and frequency.

For acoustical designing in architecture it is convenient to use an "average" absorption coefficient α which is assumed to depend only on the physical characteristics of the material and not on the sound field. These are the values of absorption that are given in this section. A surface having an absorption coefficient α and area S square feet is said to have an absorption of αS sabins. Thus the *sabin* (sometimes called a square-foot unit of absorption) is the absorption equivalent of 1 sq ft of material having an absorption coefficient of unity.

A quantity which describes the acoustical properties of a material that is more fundamental than absorption coefficient is its *acoustic impedance,* defined as the complex ratio of sound pressure to the corresponding particle velocity at the surface of the material. Because of the complexities involved in the solutions to problems of room acoustics by boundary-value theory in terms of boundary impedances,[1] the simpler concept of absorption coefficient is usually employed in calculating the acoustical properties of rooms, as indicated in the following section.

Most manufactured acoustical materials depend largely on their porosity for their acoustic absorption, the sound waves being converted into heat as they are propagated into the interstices of the material and also by vibration of the small fibers of the material. Another important mechanism of absorption is panel vibration; when sound waves force a panel into motion the resulting flexual vibration converts a fraction of the incident sound energy into heat.

The average value of absorption coefficient of a material varies with frequency. Tables usually list the values of α at 125, 250, 500, 1,000, and 4,000 cps, or at 128, 256,

[1] P. M. Morse, "Vibration and Sound," chap. VIII, McGraw-Hill Book Company, Inc., New York, 1948.

512, 1,024, 2,048, and 4,096 cps, which for practical purposes are identical. In comparing materials which are used for noise-reduction purposes in offices, banks, corridors, etc., it is sometimes useful to employ a single figure called the noise-reduction coefficient (abbreviated NRC) of the material which is the average of the absorption coefficients at 250, 500, 1,000, and 2,000 cps, to the nearest multiple of 0.05.

Figures 3j-1 through 3j-4 give the absorption coefficient vs. frequency for several types of acoustical material.[1] The absorption-frequency characteristics of regularly perforated cellulose fiber tile $\frac{3}{4}$ in. thick is shown in Fig. 3j-1. These curves represent average coefficients for materials of the same type, thickness, and method of mounting but of different manufacture. Similar data are shown in Fig. 3j-2 for fissured mineral tile $\frac{13}{16}$ in. thick. Values of noise-reduction coefficient are shown to the right of the graph. Values of absorption coefficient for various types of building materials are

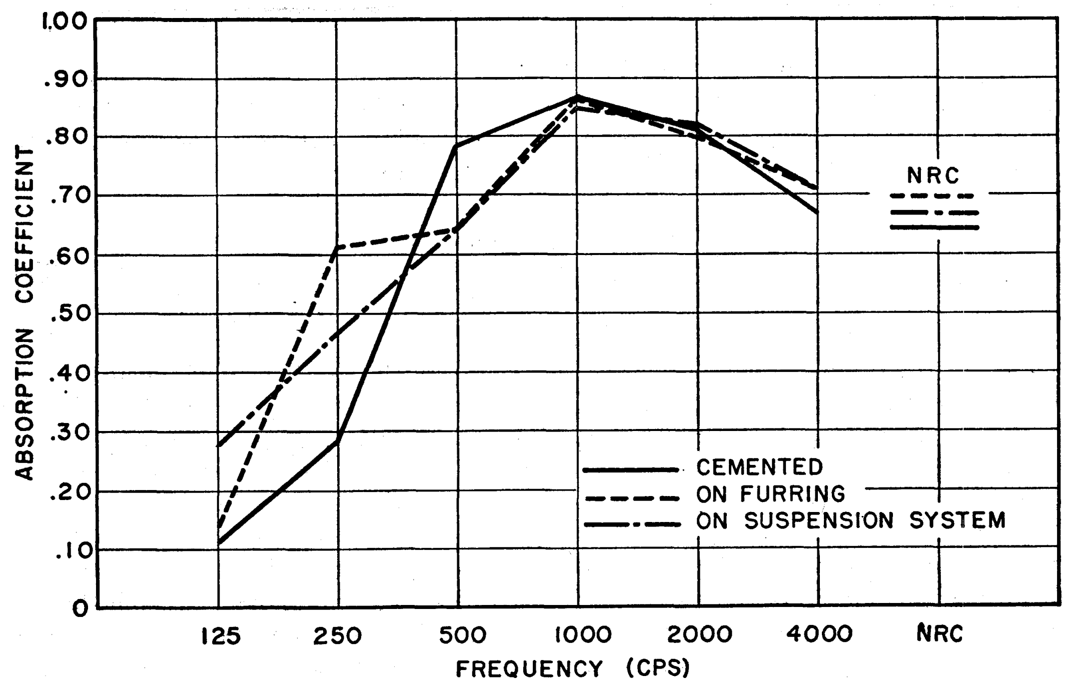

FIG. 3j-1. The absorption vs. frequency characteristic for regularly perforated cellulose fiber acoustical tile. These data represent average values for $\frac{3}{4}$-in. tile having the same thickness and mounted in the same way but of different manufacture. (*After H. J. Sabine.*)

given in Table 3j-1.[2] The equivalent absorption of individuals and seats, expressed in sabins, is given in Table 3j-2. More complete data, and data for other types of material, are given in Knudsen and Harris.[3] Sound-absorptive materials and structures may be classified in the following way: (1) prefabricated units, including acoustical tile, tile boards, and certain mechanically perforated units backed with absorptive material; (2) acoustical plasters; (3) acoustical blankets, consisting of mineral wool, glass fibers, hair felt, or wood fibers held together in blanket form by a suitable binder; (4) panel absorbers, including panels of plywood, paperboard, and pressed-wood fiber; (5) membrane absorbers consisting of a membrane of negligible stiffness backed by an enclosed air space; (6) resonator absorbers of the Helmholtz type; and (7) special types.

[1] C. M. Harris, "Handbook of Noise Control," chapter by H. J. Sabine, McGraw-Hill Book Company, Inc., New York, in preparation.

[2] Acoustical Materials Association, *Bull.* XV, New York, 1955.

[3] V. O. Knudsen and C. M. Harris, "Acoustical Designing in Architecture," John Wiley & Sons, Inc., New York, 1950.

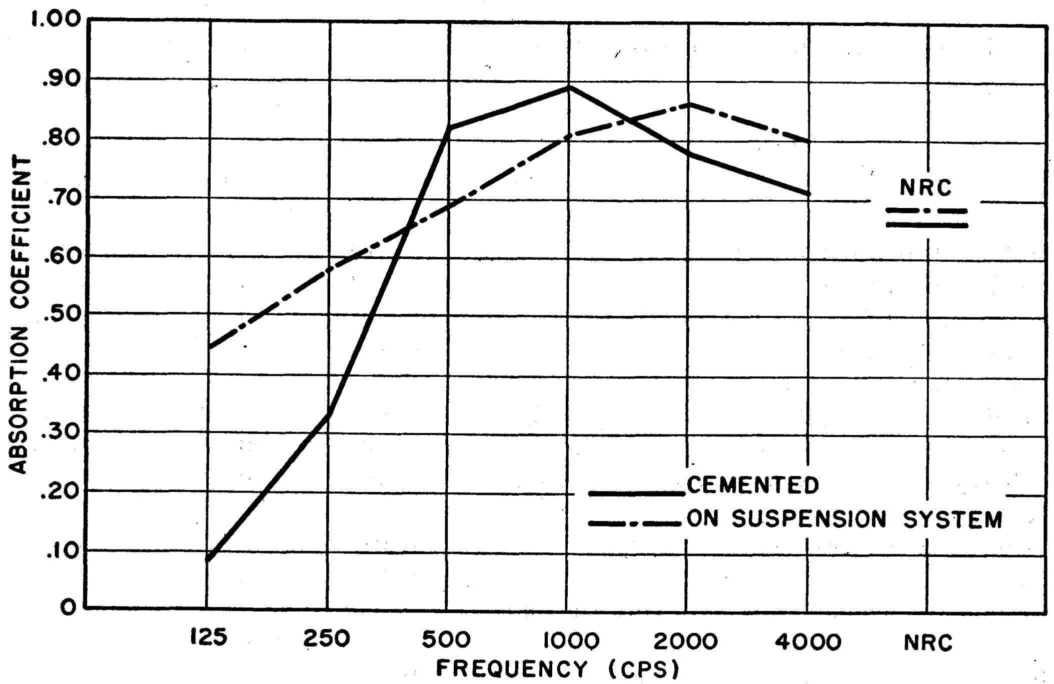

FIG. 3j-2. The absorption vs. frequency characteristic for fissured mineral tile. These data represent average values for $\frac{13}{16}$-in. tile having the same thickness and mounted in the same way but of different manufacture. (*After H. J. Sabine.*)

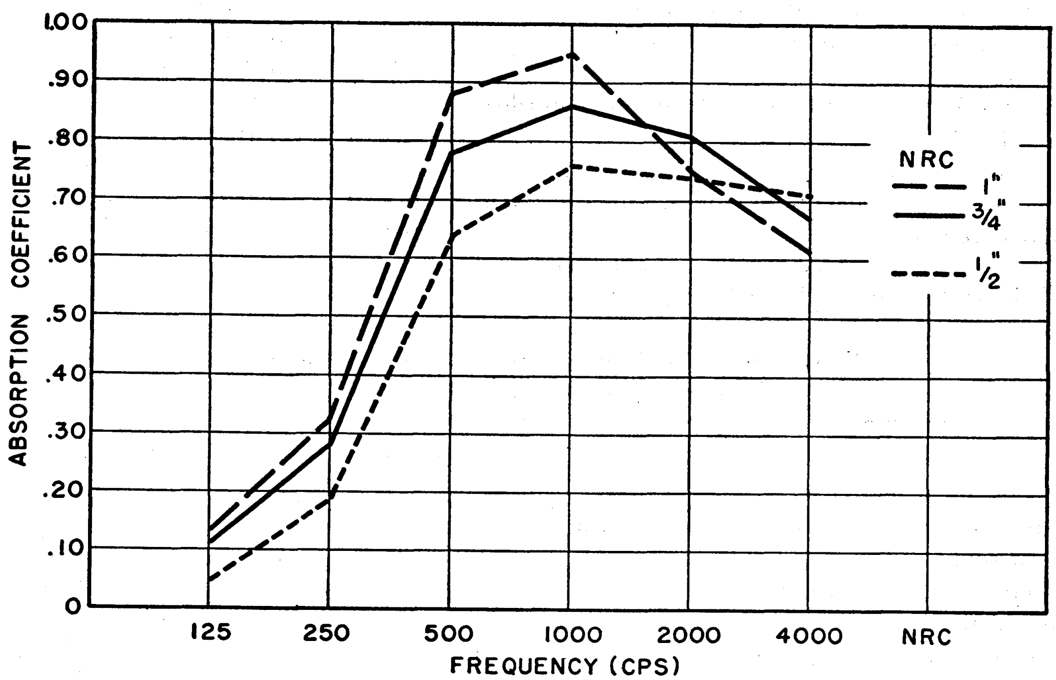

FIG. 3j-3. The absorption vs. frequency characteristic for regularly perforated cellulose fiber acoustical tile which has been spot-cemented to a rigid surface. These data represent the average value for tiles of different manufacture, mounted in the same way and having different thickness. (*After H. J. Sabine.*)

3j-2. Reverberation-time Calculations. After sound has been produced in or enters an enclosed space it will be reflected by the boundaries of the enclosure. Although some energy is lost at each reflection, several seconds may elapse before the sound decays to inaudibility. This prolongation of sound after the original source has stopped is called *reverberation*, a certain amount of which is found to add a pleasing

TABLE 3j-1. ABSORPTION COEFFICIENTS FOR BUILDING MATERIALS*

Material	Coefficients		
	125 cps	500 cps	2,000 cps
Brick wall, painted	0.012	0.017	0.023
Same, unpainted	0.024	0.03	0.049
Carpet, unlined	0.09	0.20	0.27
Same, felt-lined	0.11	0.37	0.27
Fabrics, hung straight:			
Light, 10 oz/sq yd	0.04	0.11	0.30
Medium, 14 oz/sq yd	0.06	0.13	0.40
Heavy, draped, 18 oz/sq yd	0.10	0.50	0.82
Floors:			
Concrete or terrazzo	0.01	0.015	0.02
Wood	0.05	0.03	0.03
Linoleum, asphalt, rubber or cork tile on concrete		0.03–0.08	
Glass	0.035	0.027	0.02
Marble or glazed tile	0.01	0.01	0.015
Openings:			
Stage, depending on furnishings		0.25–0.75	
Deep balcony, upholstered seats		0.50–1.00	
Grills, ventilating		0.15–0.50	
Plaster, gypsum, or lime, smooth finish on tile or brick	0.013	0.025	0.04
Same, on lath	0.02	0.03	0.04
Plaster, gypsum, or lime, rough finish on lath	0.039	0.06	0.054
Wood paneling	0.08	0.06	0.06

* From *AMA Bull.* XV, no. 2.

TABLE 3j-2. ABSORPTION OF SEATS AND AUDIENCE*
(In sabins per person or unit of seating)

	125 cps	500 cps	2,000 cps
Audience, seated, depending on character of seats, etc	1.0–2.0	3.0–4.3	3.5–6.0
Chairs, metal or wood	0.15	0.17	0.20
Wood pews		0.40	
Pew cushions (without pews)	0.75–1.1	1.45–1.90	1.4–1.7
Theater and auditorium chairs:			
Wood-veneer seat and back		0.25	
Upholstered in leatherette		1.6	
Heavily upholstered in plush or mohair		2.6–3.0	

* From *AMA Bull.* XV, no. 2.

characteristic to the acoustical qualities of a room. On the other hand, excessive reverberation can ruin the acoustical properties of an otherwise well-designed room.

Because of the importance of the proper control of reverberation in rooms, a standard of measure called *reverberation time* (abbreviated t_{60}) has been established. It is one of the important parameters in architectural acoustics. This is the time required for a specified sound to die away to one-thousandth of its initial pressure, a drop in sound pressure level of 60 db. It is given by the following equation:

$$t_{60} = \frac{0.049V}{S[-2.30 \log_{10} (1 - \bar{\alpha})] + 4mV} \quad \text{sec} \tag{3j-2}$$

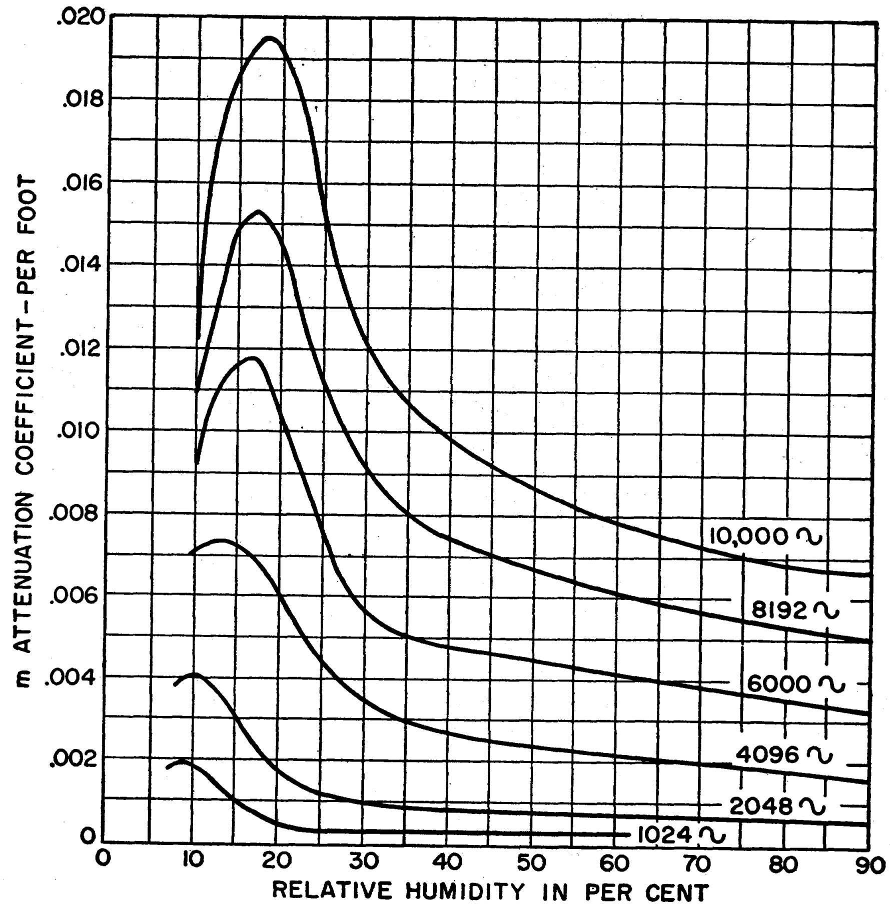

FIG. 3j-4. Values of the attenuation coefficient m as a function of relative humidity for different frequencies. (*After V. O. Knudsen and C. M. Harris.*)

and when $\bar{\alpha}$ is small compared with unity,

$$t_{60} = \frac{0.049V}{S\bar{\alpha} + 4mV} \quad \text{sec} \tag{3j-3}$$

where V = volume of the room, cu ft

S = total surface area, sq ft

$\bar{\alpha}$ = average absorption coefficient given by

$$\bar{\alpha} = \frac{\alpha_1 S_1 + \alpha_2 S_2 + \alpha_3 S_3 + \cdots}{S_1 + S_2 + S_3 + \cdots} = \frac{a}{S} \tag{3j-4}$$

α_1 = absorption coefficient of area S_1, etc.

a = total absorption in the room, sabins

The quantity m is the attenuation coefficient for air given by Fig. 3j-4.[1] For relatively small auditoriums and frequencies below 2,000 cps, the mV term can usually be neglected so that Eq. (3j-3) reduces to

$$t_{60} = \frac{0.049V}{S\bar{\alpha}} \quad \text{sec} \tag{3j-5}$$

3j-3. Optimum Reverberation Time. A certain amount of reverberation in a room adds a pleasing quality to music. Since the reverberation time one would consider to be optimum is a matter of personal preference, it is not a quantity that can be calculated from a formula. On the other hand, useful engineering-design data may be obtained from a critical evaluation of empirical data based upon the preference evaluations of large groups of individuals. The results of such information from all available sources considered reliable, in this country and abroad, have been carefully

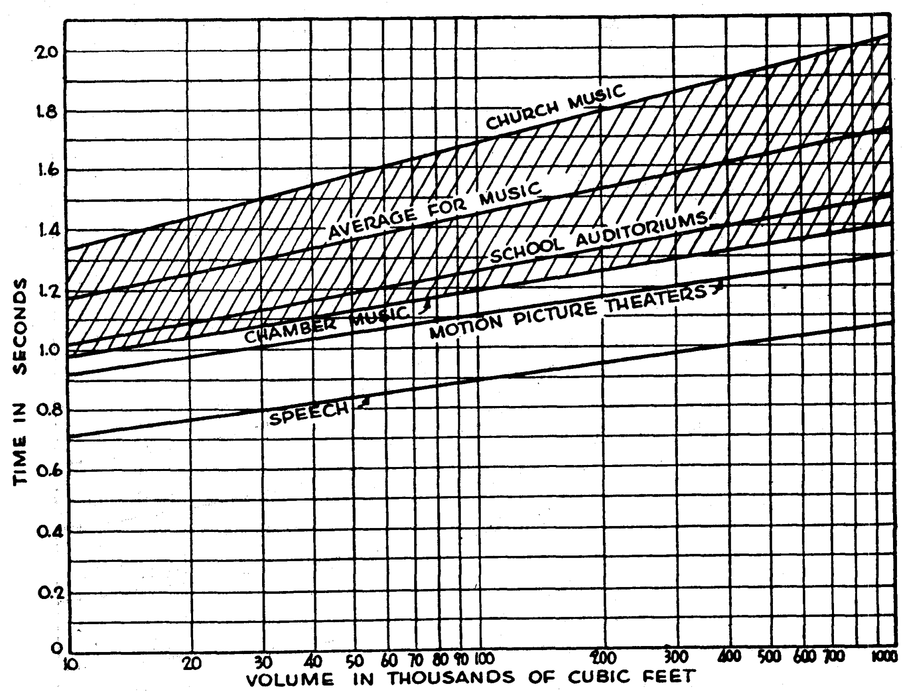

FIG. 3j-5. Optimum reverberation time at 512 cps for different types of rooms as a function of room volume. This figure should be used in conjunction with Fig. 3j-6 to obtain optimum reverberation time as a function of frequency. (*After V. O. Knudsen and C. M. Harris.*)

evaluated by Knudsen and Harris,[1] who have published the curves for optimum reverberation time shown in Figs. 3j-5 and 3j-6. The data in Fig. 3j-5 give the optimum reverberation times at 512 cps as a function of volume for rooms and auditoriums that are used for different purposes. Since the optimum reverberation time for music depends on the type of music, it is represented by a broad band. The optimum reverberation time for a room used primarily for speech is considerably shorter; a reverberation time longer than those shown results in a decrease in speech intelligibility.

The optimum reverberation times at frequencies other than 512 cps is obtained by multiplying the values given in Fig. 3j-5 by the ratio R from Fig. 3j-6 for the desired frequency. These data indicate that below 512 cps the optimum reverberation time may fall anywhere in a wide range shown by the crosshatched band; smaller rooms usually have preferred ratios that are in the lower part of the band.

[1] *Ibid.*

3j-4. Air-borne Sound Transmission through Partitions. The fraction of incident sound energy transmitted through a partition is called its transmission coefficient τ. In rating the noise-insulating value of partitions, windows, and doors, it is generally convenient to employ a logarithmic quantity, transmission loss T.L., which is equal to the number of decibels by which sound energy that is incident on a partition is reduced in transmission through it. The two quantities are related by the equation

$$\text{T.L.} = 10 \log \frac{1}{\tau} \quad \text{db} \tag{3j-6}$$

Air-borne sound is transmitted through a so-called "rigid" partition, such as a wall of concrete or brick, by forcing it into vibration; then the vibrating partition becomes a secondary source, radiating sound to the side opposite the original source. Over a large portion of the audible range, such a partition, on the average, approximates a mass-controlled system so that its transmission loss should increase 6 db each time the weight of the partition is doubled. In most actual partitions the increase is usually less, say 4 to 5 db for the average frequency range between 128 and 2,048 cps.

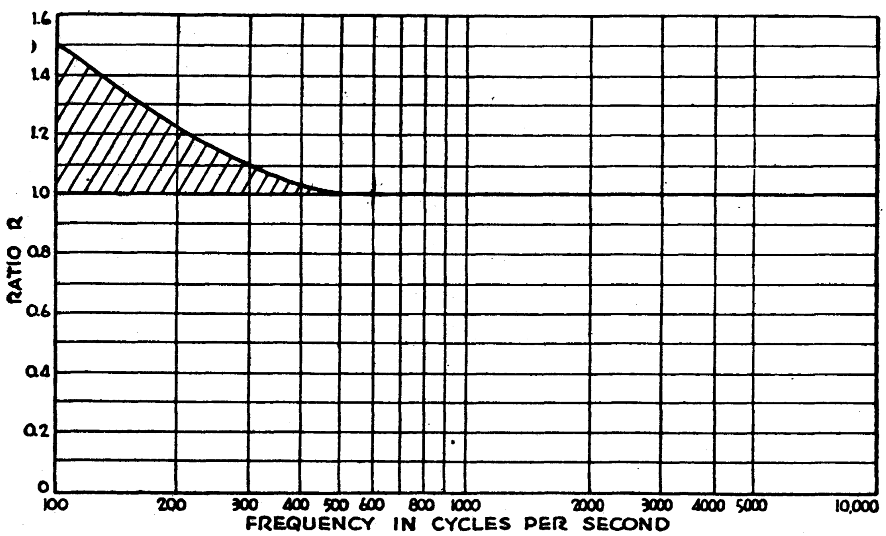

FIG. 3j-6. Chart for computing optimum reverberation time as a function of frequency The time at any frequency is given in terms of a ratio R which should be multiplied by the optimum time at 512 cps (from Fig. 3j-5) to obtain the optimum time at that frequency. (*After V. O. Knudsen and C. M. Harris.*)

This is illustrated by Fig. 3j-7, which gives the transmission loss (averaged over frequency in the range from 128 to 4,096 cps) as a function of weight of the partition in pounds per square foot of surface area. The straight line represents the calculated transmission loss assuming that the values of T.L. increase 6 db for each doubling of the weight. The transmission loss for a partition is not constant with frequency, increasing usually 3 to 6 db/octave.

Note that a compound-wall construction can yield relatively high sound insulation with relatively low mass per unit wall area. The double-wall construction is one such example. It is important that the separation between the walls be as complete as possible—structural ties will greatly reduce the effectiveness of such a structure.

Values of transmission loss for various types of walls and floors employed in ordinary building construction are given in Table 3j-3 and Fig. 3j-7.[1]

3j-5. Noise Level within a Room. The sound level of noise which is transmitted into a room from the outside depends on (1) the noise-insulating properties of its

[1] Knudsen and Harris, *op. cit.;* Sound Insulation of Wall and Floor Constructions, Building Materials and Structures, *Natl. Bur. Standards* (*U.S.*) *Rept.* 144 (1955).

bounding surfaces, (2) the total absorption in the room, and (3) the characteristics of the noise source. The following formula gives a rating of the over-all noise reduction provided by the enclosure. It represents, approximately, the difference between the noise level outside a room and the noise level inside a room.

$$\text{Noise-insulation factor} = 10 \log \frac{a}{T} \quad \text{db} \tag{3j-7}$$

where a represents the total absorption in the room in sabins defined by Eq. (3j-4), and T represents the total transmittance of the enclosure given by

$$T = \tau_1 S_1 + \tau_2 S_2 + \tau_3 S_3 + \cdots \tag{3j-8}$$

where τ_1 is equal to the transmission coefficient of area S_1, etc.

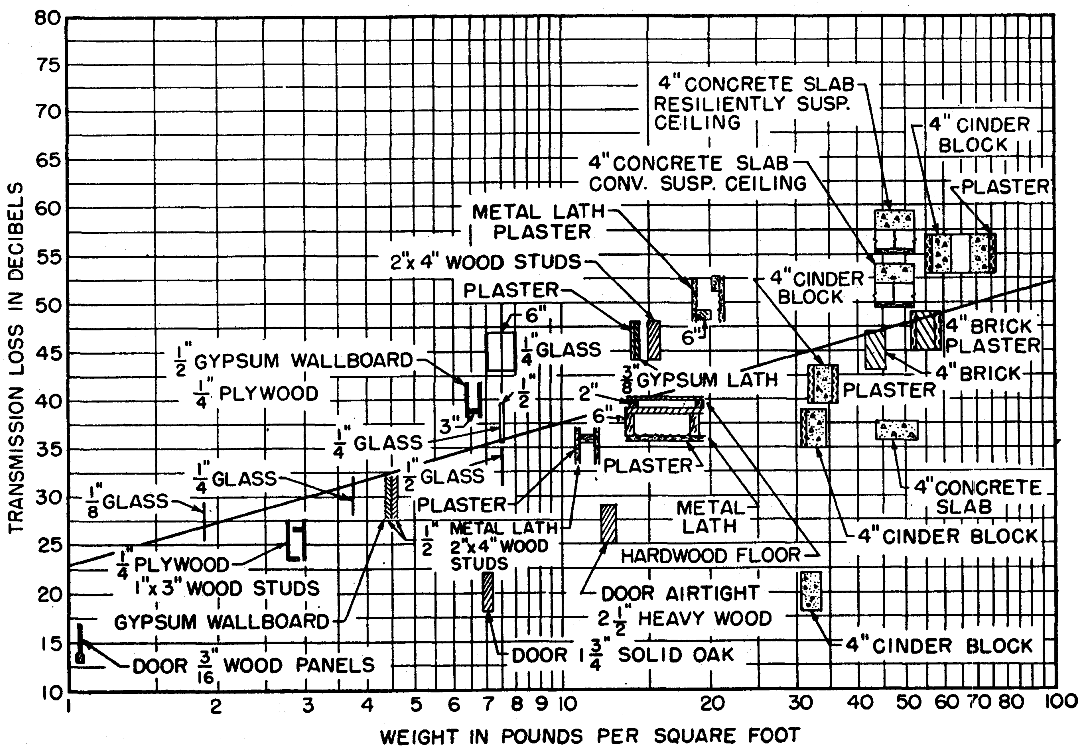

FIG. 3j-7. Transmission loss, average over frequency in the range from 128 to 4,096 cps, as a function of weight of the parts in pounds per square foot of surface area.

If a source of noise is within a room, then at distances near to the source the sound pressure decreases inversely with increasing distance from the source; there is a decrease in sound pressure level of 6 db for each doubling of the distance from the source, just as if the source were in the open air. However, at every point in the room there will be an additional contribution to the total pressure as a result of reflections from the walls. As one recedes from the source the reflected contributions become more and more important until direct sound from the source becomes negligible by comparison.

Then if the sound field is diffuse (perfect diffusion is said to exist if the sound pressure everywhere in the room is the same, and it is equally probable that the waves are traveling in every direction) the sound pressure level in the room will be given approximately by

$$L_p = 10 \log \frac{W}{a} + 136.4 \quad \text{db} \tag{3j-9}$$

if a value of $\rho c = 40.8$ rayls is assumed for air, and

where W = power of the sound source, watts
a = total absorption of the room, sabins

A consideration of the above formula shows that, if the acoustic-power output of the noise source remains constant, and if the total absorption in the room is increased from a_1 to a_2 the reduction in noise level is given by

$$\text{Noise reduction} = 10 \log \frac{a_2}{a_1} \quad \text{db} \tag{3j-10}$$

TABLE 3j-3. VALUES OF TRANSMISSION LOSS T.L. VS. FREQUENCY FOR VARIOUS TYPES OF WALL AND FLOOR CONSTRUCTION*

Construction	Weight, lb/sq ft	Average, 128–4,096	128 cps	256 cps	512 cps	1,024 cps	2,048 cps	4,096 cps	Authority†
Wood studs 2 by 4 in., 16 in. o.c.: With lime plaster $\frac{7}{8}$ in. thick on metal lath	19.8	44	26	41	44	52	56	58	N.B.S.
With gypsum plaster $\frac{1}{2}$ in. thick on $\frac{3}{4}$-in. gypsum lath	15.2	41	33	31	39	46	49	66	N.B.S.
Wood studs, 2 by 4 in., staggered; $\frac{7}{8}$-in. gypsum plaster on metal lath	19.8	50	44	47	47	50	52	63	N.B.S.
Staggered wood studs 1 by 3 in., $\frac{1}{4}$-in. plywood glued to both sides	2.6	26	14	20	28	33	40	30	N.B.S.
Two sets of 2- by 2-in. wood studs, $\frac{1}{4}$-in. plywood sheet inserted in $\frac{1}{4}$-in. space between studs, $\frac{1}{4}$-in. plywood faces, slightly compressed paper-backed mineral wood inserted in both air spaces, total panel thickness $4\frac{3}{4}$ in.	5.1	37	20	31	37	41	49	50	N.B.S.
Steel studs, 3 in., 16 in. o.c., $\frac{7}{8}$-in. gypsum plaster on expanded metal lath	19.6	37	30	28	35	40	43	53	N.B.S.
Brick, laid on edge; gypsum plaster on both sides	31.6	42		40	37	49	59		N.B.S.
Tile, hollow partition, 4 in. thick, pumice-cement block, two cells 4 by 8 by 16 in., no plaster	15.5	11	8	5	9	14	19	17	N.B.S.
Same, but one side plastered	20.4	35	31	27	35	36	40	47	N.B.S.
Same, but both sides plastered	25.3	37	32	34	36	39	42	52	N.B.S.
Cinder block, hollow partition 3 by 8 by 16 in., plaster on both sides	32.2	45	34	37	42	51	57	64	N.B.S.
Multiple-block partition; two leaves, each of 3-in. hollow blocks, separated by 2-in. cavity and built on opposite sides of gap separating rooms; outer faces plastered (two partitions of nominally the same construction)	28	9	54	38	47	49	69	77	N.P.L.
Wood joints, 2 by 8 in., $\frac{1}{2}$-in. fiberboard lath and $\frac{1}{2}$-in. gypsum plaster ceiling; 1-in. pine subflooring and 1-in. pine finish flooring	14.3	45	23	34	47	55	54	69	N.B.S.
Same joists and ceiling as above; 1-in. pine subfloor; $\frac{1}{2}$-in. fiberboard, 1- by 3-in. sleepers, and 1-in. pine finish floor	16.2	50	30	37	50	57	65	79	N.B.S.

* For the average values for other types of construction, and for windows and doors, also see Fig. 3j-7.
† N.B.S. denotes National Bureau of Standards; N.P.L. denotes National Physical Laboratory.

According to this equation, which should be regarded as an engineering approximation to actual conditions, if the absorption in a room is increased by a factor of 4 the noise reduction will be 6 db. It shows that the addition of absorption level in a room will provide substantial noise reduction in average level in a room that is relatively bare but little decrease level in a highly damped room. The reduction will be different at different frequencies since the total absorption is a function of frequency. However, it is sometimes convenient to employ the noise-reduction coefficient of a material to obtain a single noise-reduction figure. Besides reducing the steady-state level, the addition of absorptive treatment in a room also provides beneficial effects by reducing the reverberation time in the room and by localizing the source of noise to the area in which it originates—thereby minimizing unexpected noises.

TABLE 3j-4. RECOMMENDED ACCEPTABLE AVERAGE NOISE LEVELS IN UNOCCUPIED ROOMS*

	Decibels
Radio, recording, and television studios	25–30
Music rooms	30–35
Legitimate theaters	30–35
Hospitals	35–40
Motion-picture theaters, auditoriums	35–40
Churches	35–40
Apartments, hotels, homes	35–45
Classrooms, lecture rooms	35–40
Conference rooms, small offices	40–45
Courtrooms	40–45
Private offices	40–45
Libraries	40–45
Large public offices, banks, stores, etc	45–55
Restaurants	50–55

The levels given in this table are "weighted"; i.e., they are the levels measured with a standard sound-level meter incorporating an "A" (40-db) frequency-weighting network.

* V. O. Knudsen and C. M. Harris, "Acoustical Designing in Architecture," John Wiley & Sons, Inc., New York, 1950.

3j-6. Acceptable Noise Levels for Various Types of Room. Table 3j-4 gives values of recommended acceptable average noise levels for unoccupied rooms with the ventilation system in operation. These values are used for design purposes, for example, in computing the amount of over-all noise insulation that should be provided for a room. They hold for typical room-noise spectra. Although even lower noise levels than those which are listed may provide some advantage under certain circumstances, and may be desirable if cost is not a factor, this table gives values which represent a combination of acceptability and economic practicality. For certain types of room the values which are recommended are lower than those which are commonly found.

3k. Speech and Hearing

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The data concerning hearing are, without exception, empirical in derivation. Consequently, the values reported always represent some parameter of a population, most often a mean, and the reader is warned to bear constantly in mind the many sources of variability that attach to any particular measurement.

3k-1. Physical Dimensions of the Ear

TABLE 3k-1. PHYSICAL DIMENSIONS OF THE EAR*

Pinna:
- Mean length, young men, 65.0 mm
- Range, 52–79 mm

Auditory meatus:
- Cross section, 0.3–0.5 cm^2
- Diameter, 0.7 cm
- Length, 2.7 cm
- Volume, 1.0 cc

Tympanic membrane:
- Area, 0.5–0.9 cm^2 (roughly circular)
- Thickness, about 0.1 mm
- Volume elasticity for 10 cps, equivalent to about 8 cc air
- Displacement amplitude for 1,000-cps tone (at threshold), 10^{-9} cm
- Displacement amplitude for low-frequency tones (threshold of feeling), about 10^{-2} cm

Middle ear:
- Total volume, about 2 cc
- Malleus:
  - Weight, 23 mg
  - Length, 5.5–6.0 mm
- Incus: weight, 27 mg
- Stapes:
  - Weight, 215 mg
  - Length of footplate, 3.2 mm
  - Width of footplate, 1.4 mm
  - Area of footplate, 3.2 mm^2
  - Width of elastic ligament, 0.015–0.1 mm

Cochlea:
- Length of cochlear channels, 35 mm
- Height of scala vestibuli or scala tympani, about 1 mm (great variability)
- Round window: area, 2 mm^2
- Basilar membrane:
  - Width at stapes, 0.04 mm
  - Width at helicotrema, 0.5 mm
- Helicotrema: area of opening, 0.25–0.4 mm^2

* S. S. Stevens, ed., "Handbook of Experimental Psychology," John Wiley & Sons, Inc., New York, 1951.

3k-2. Acoustic Impedance of the Ear. Reasonable agreement on measurements below 1,000 cps has been obtained. The reference point for measurements is just

[1] This section benefited from the advice and assistance of Dr. S. S. Stevens and Mrs. Nancy C. Waugh.

TABLE 3k-2. ACOUSTIC IMPEDANCE OF THE EAR IN ACOUSTIC OHMS, MEASURED JUST WITHIN THE MEATUS

Frequency	Total impedance	Resistive component	Reactive component
250	200	50	−190
350	150	40	−145
500	125	35	−115
700	70	25	−65
1,000	55	25	−50

Above 1,000 cycles, measurements depend increasingly on the method of measurement.

TABLE 3k-3. MINIMUM AUDIBLE PRESSURE AT ENTRANCE TO EXTERNAL EAR CANAL (MAC), IN DECIBELS SPL

	Frequency									
	80	125	250	500	1,000	2,000	4,000	6,000	8,000	10,000
Threshold....	43.5	30.0	18.5	11.5	9.0	8.0	9.5	13.0	17.0	21.0

The following corrections may be applied if it is desired to find thresholds for other conditions:

a. MAC to Threshold Pressure at Eardrum[a]

	Frequency								
	125	250	500	1,000	2,000	4,000	6,000	8,000	10,000
Add........	0.0	0.0	−0.5	−1.0	−4.5	−10.5	−4.0	−2.5	

b. MAC to Equivalent Coupler Calibration of Various Earphones[a,b,c]

	Frequency									
	125	250	500	1,000	2,000	4,000	6,000	8,000	10,000	Coupler
Add for PDR-8 with MX-41/AR	+8.0	+4.0	+0.5	+1.0	+1.5	+4.5	+12.0	−3.0	..	NBS-9A
Add for WE 705A.......	+13.0	+4.0	+0.5	+1.0	+4.5	+5.0	−0.5	−7.0	..	NBS-9A
Add for STC 4026A......	+14.5	+11.0	+0.5	−3.5	+1.0	+0.5	−4.0	−7.5	..	NPL[b]

[a] F. M. Wiener and D. A. Ross, The Pressure Distribution in the Auditory Canal in a Progressive Sound Field, *J. Acoust. Soc. Am.* **18,** 401–408 (1946).

[b] L. J. Wheeler and E. D. D. Dickson, The Determination of the Threshold of Hearing, *J. Laryngol. Otol.* **66,** 379–395 (1952).

[c] E. L. R. Corliss, R. F. Brown, Jr., M. D. Burkhard, R. P. Thompson, Jr., and J. F. Mullen, Methods for Calibration of Hearing Diagnostic Instruments, *Natl. Bur. Standards* (*U.S.*) *Rept.* 1470, 1–43 (1952).

TABLE 3k-3. MINIMUM AUDIBLE PRESSURE AT ENTRANCE TO EXTERNAL EAR CANAL (MAC), IN DECIBELS SPL (*Continued*)

c. MAC to Free Field (MAF) (plane wave, 0° azimuth in absence of head)[d]

	Frequency								
	125	250	500	1,000	2,000	4,000	6,000	8,000	10,000
Add	+1.0	+0.5	−2.0	−4.0	−11.0	−12.5	−7.0	−3.0	−3.0

d. Mean Monaural to Mean Binaural Listening[e]

	Frequency				
	125–2,000	4,000	6,000	8,000	10,000
Add	−2.0	−3.0	−4.0	−5.0	−6.0

e. Reference Age Group (18–25) to Older Age Groups[f]

	Frequency					
	125–1,000	2,000	4,000	6,000	8,000	10,000
Add for:						
Men 30–39	+1.0	+2.0	+5.0	+6.0	+6.0	+7.0
Men 40–49	+2.0	+5.0	+13.0	+13.0	+11.0	+13.0
Men 50–59	+5.0	+13.0	+27.0	+32.0	+35.0	+35.0
Women 30–39	+1.0	+2.0	+3.0	+4.0	+4.0	+4.0
Women 40–49	+3.0	+5.0	+6.0	+8.0	+9.0	+9.0
Women 50–59	+5.0	+9.0	+13.0	+18.0	+20.0	+22.0

[d] L. J. Sivian and S. D. White, On Minimum Audible Sound Fields, *J. Acoust. Soc. Am.* **4,** 288–321 (1933).

[e] H. Fletcher, "Speech and Hearing in Communication," p. 131, D. Van Nostrand Company, Inc., New York, 1953.

[f] J. C. Steinberg, H. C. Montgomery and M. B. Gardner, Results of the World's Fair Hearing Tests, *J. Acoust. Soc. Am.* **12,** 291–301 (1940); J. C. Webster, H. W. Himes, and M. Lichtenstein, San Diego County Fair Hearing Survey, *J. Acoust. Soc. Am.* **22,** 473–483 (1950).

within the external meatus. The values in Table 3k-2 are representative but are subject to wide variations among individuals.[1]

3k-3. Minimum Audible Sound. The best recent measurements use as their point of reference the sound pressure level of a tone, heard one-half the time, and measured at the entrance to the external meatus. The observations were made on healthy young men, eighteen to twenty-five years of age, tested individually with earphones, one ear at a time. Sound pressures were determined with a probe-tube microphone and are given in decibels above 0.0002 dyne/cm^2. $N = 1{,}200$ ears.[2]

[1] E. Waetzmann and L. Keibs, Hörschwellenbestimmungen mit dem Thermophon und Messungen am Trommelfell, *Ann. Physik* **26,** 141–144 (1936); O. Metz, The Acoustic Impedance Measured on Normal and Pathological Ears, *Acta Oto-Laryngol., Suppl.* 63, 1–254 (1946); A. H. Inglis, C. H. G. Gray, and R. T. Jenkins, A Voice and Ear for Telephone Measurements, *Bell System Tech. J.* **11,** 293–317 (1932).

[2] R. S. Dadson and J. H. King, A Determination of the Normal Threshold of Hearing and Its Relation to the Standardization of Audiometers, *J. Laryngol. Otol.* **66,** 366–378 (1952); L. J. Wheeler and E. D. D. Dickson, The Determination of the Threshold of Hearing, *J. Laryngol. Otol.* **66,** 379–395 (1952).

3k-4. Threshold of Feeling or Discomfort. The upper limit for a tolerable intensity of sound rises substantially with increasing habituation. Moreover, a variety of subjective effects are reported, such as discomfort, tickle, pressure, and pain, each at a slightly different level. As a simple engineering estimate it can be said that naïve listeners reach a limit at about 125 db SPL and experienced listeners at 135 to 140 db. These are over-all measures of sound falling within the audible range and are roughly independent of frequency.

3k-5. Differential Thresholds for Pure Tones and Noise. A differential threshold represents a careful determination by laboratory methods of the ability of a subject to just detect, and report, a difference in any specific property of a sound, all other factors presumably being held constant.

TABLE 3k-4. DIFFERENTIAL THRESHOLD FOR INTENSITY, IN DECIBELS

Sensation level, db above absolute threshold	Pure tones, frequency in cps							White noise
	35	70	200	1,000	4,000	7,000	10,000	
5	....	....	4.75	3.03	2.48	4.05	4.72	1.80
10	7.24	4.22	3.44	2.35	1.70	2.83	3.34	1.20
20	4.31	2.38	1.93	1.46	0.97	1.49	1.70	0.47
30	2.72	1.52	1.24	1.00	0.68	0.90	1.10	0.44
40	1.76	1.04	0.86	0.72	0.49	0.68	0.86	0.42
50	....	0.75	0.68	0.53	0.41	0.61	0.75	0.41
60	....	0.61	0.53	0.41	0.29	0.53	0.68	0.41
70	....	0.57	0.45	0.33	0.25	0.49	0.61	
80	....	....	0.41	0.29	0.25	0.45	0.57	
90	....	....	0.41	0.29	0.21	0.41		
100	....	....	....	0.25	0.21			
110	....	....	....	0.25				

TABLE 3k-5. DIFFERENTIAL THRESHOLD FOR FREQUENCY, IN $\Delta F/F$*

Sensation level, db above absolute threshold	Pure tones, frequency in cps						
	60	125	250	500	1,000	2,000	4,000
5	0.0252	0.0110	0.0097	0.0065	0.0049	0.0040	0.0077
10	0.0140	0.0060	0.0053	0.0035	0.0027	0.0022	0.0042
15	0.0092	0.0040	0.0035	0.0024	0.0018	0.0014	0.0028
20	0.0073	0.0032	0.0028	0.0019	0.0014	0.0012	0.0022
30		0.0032	0.0028	0.0019	0.0014	0.0011	0.0022

* J. D. Harris, Pitch Discrimination, *J. Acoust. Soc. Am.* **24,** 750–755 (1952).

The method for determining the differential threshold for intensity of pure tones employed one tone beating with a second tone at 3 beats per second.[1] Much evidence is available to support what should be kept always in mind, that thresholds determined

[1] R. R. Reisz, Differential Intensity Sensitivity of the Ear for Pure Tones, *Phys. Rev.* **31,** 867–875 (1928).

by other methods are a function of numerous psychological parameters and will differ systematically from the values in Table 3k-4. A more conventional method was used to determine the thresholds for white noise, with the results given in the last column.[1]

The ability to distinguish pitch is subject to a greater range of individual variability than other functions reported here. The data given are for three trained listeners and have been smoothed in both directions. Untrained listeners usually require a greater frequency difference than that reported here. Note also that individual listeners commonly show idiosyncrasies at particular frequencies.

3k-6. Masking. Masking refers to our inability to hear a weak sound in the presence of a louder sound. It is usually measured by the amount of change in the threshold of the weaker sound, i.e., how much more intense must the weak sound be made in order to be heard over the masking sound, than it needed to be when the masking sound was not present. The masking of one pure tone by another is a complex function of the particular frequencies and of the absolute level of the respective tones. See any standard text on hearing for the curves describing this relationship.

The masking of a pure tone by a noise with a reasonably flat and continuous spectrum is a linear function (except at levels below 10 db) of the total intensity within a "critical band" centered on the masked tone. The width of the critical band of frequencies whose total energy is just equal to the energy of the masked tone is given by Table 3k-6.

TABLE 3k-6. WIDTH OF "CRITICAL BAND" ΔF AS A FUNCTION OF CENTER FREQUENCY F (10 log ΔF)*

	Frequency							
	100	250	500	1,000	2,000	4,000	8,000	10,000
ΔF, db	19.4	17.1	17.1	18.0	19.9	23.1	27.7	29.2

* N. R. French and J. C. Steinberg, Factors Governing the Intelligibility of Speech Sounds, *J. Acoust. Soc. Am.* **19**, 90–119 (1947).

The masking of one continuous noise by another can be thought of as a case of differential sensitivity to change in the intensity of a noise (see last column of Table 3k-4). Thus, above 40 db SPL, if a weak noise is more than 10 db less intense than a very similar masking noise, the weak noise will not be heard; its presence or absence does not produce a discriminable difference in intensity. If the spectral composition of the two noises, masking and masked, are quite different, then the critical-band concept must be employed.

3k-7. Sounds of Short Duration. Acoustic disturbances of very short duration, i.e., less than 0.0001 sec, are heard only to the extent that they transmit energy to the ear. Short pulses at ultrasonic frequencies are generally not heard unless they are rectified. Impulse or step functions excite the ear, but not efficiently.

At the opposite extreme, tones, or continuous noise, of duration greater than from 0.2 to 0.5 sec, are generally heard independently of duration. Between these limits relatively complex relations are found.[2]

As a first approximation for both tones and noise, the effective intensity of short sounds is a function of total energy integrated over the duration of the sound. More

[1] G. A. Miller, Sensitivity to Changes in the Intensity of White Noise and Its Relation to Masking and Loudness, *J. Acoust. Soc. Am.* **19**, 609–619 (1947).

[2] S. S. Stevens, ed., "Handbook of Experimental Psychology," pp. 1020–1021, John Wiley & Sons, Inc., New York, 1951.

accurately, the threshold is defined by[1]

$$I_t = kIt^{0.8} \tag{3k-1}$$

For some short tones and for many types of impulse noise, account must be taken of the frequency distribution of energy. Inasmuch as the ear varies in sensitivity as a function of frequency, any change in the shape or duration of a short acoustic pulse will also change its effectiveness because of the altered spectral composition.

3k-8. Loudness. Loudness and pitch are ways in which a listener reacts to sounds. Furthermore, within limits, a listener can use numbers to describe how much of a response he makes to the sound. These numbers usefully describe how loud, or how high in pitch, a sound seems to be. It is then necessary to relate how loud it is (subjective response) to how intense it is in physical terms. The loudness of a pure tone of 1,000 cps is described by the following relationship:

$$\log L = 0.0301N - 1.204 \tag{3k-2}$$

in which L is the loudness measured in sones and N is the loudness level in phons (equal to the sound pressure level of the tone in decibels above 0.0002 dyne/cm²).[2] Another way of putting this is to say that loudness doubles for each 10-db change in sound pressure level.

TABLE 3k-7. LOUDNESS LEVEL AS A FUNCTION OF SOUND PRESSURE LEVEL AND FREQUENCY*

Sound pressure level	Frequency							
	125	250	500	1,000	2,000	4,000	8,000	10,000
10	.....	.....	.....	10.0	18.0	18.0		
20	.....	6.3	16.0	20.0	28.0	28.0	11.0	
30	4.0	18.0	26.5	30.0	37.0	36.5	20.5	17.0
40	17.0	31.0	38.5	40.0	45.5	45.0	29.5	26.0
50	34.0	45.5	52.0	50.0	55.0	54.0	38.0	35.0
60	52.0	59.5	64.5	60.0	64.0	63.5	47.0	43.5
70	70.0	72.5	76.0	70.0	73.5	72.5	56.0	53.5
80	86.0	84.5	86.0	80.0	84.5	83.0	66.0	63.5
90	98.0	95.5	96.0	90.0	95.0	94.5	77.0	73.5
100	108.0	105.5	105.0	100.0	106.0	106.0	88.0	85.5
110	118.0	115.5	113.0	110.0	117.0	117.5	101.5	98.0

* American Standard for Noise Measurement, ASA Z24.2—1942.

There is some evidence that the loudness of a noise grows more rapidly than that of a tone with an increase in sound pressure level, especially at low levels. The exact relations are less well known than those for a tone.

The loudness of tones at other frequencies than 1,000 cps is given by determining the loudness level in the manner described below and converting to tones by Eq. (3k-2).

3k-9. Loudness Level. The loudness level of a tone of 1,000 cps, expressed in phons, is defined as the sound pressure level in decibels above the reference level of 0.0002 dyne/cm².

The loudness level of tones of other frequencies is given by the empirical relations in Table 3k-7.

[1] D. B. Yntema, "The Probability of Hearing a Short Tone Near Threshold," Ph.D. Dissertation, Harvard University, 1954, 43 pp.

[2] S. S. Stevens, The Measurement of Loudness, *J. Acoust. Soc. Am.* **27**, 815–829 (1955).

Note that this table is based on the ASA standard and presumes the "free-field" measurement of sound pressure. This requires a measurement of a plane progressive wave at the listener's position before the listener is placed in the field. More meaningful measurements would doubtless be obtained from pressure measurements at the ear. For this purpose, apply the corrections contained in Table 3k-3*c* to the ear canal pressures before entering Table 3k-7.

To enter the table with sound pressure levels measured under other conditions, first add the corrections in Table 3k-3*c*, then subtract rather than adding corrections in Tables 3k-3*a* through 3k-3*d*. Note, however, that corrections given for presbycusis in Table 3k-3*e* may give quite misleading results because of recruitment at high frequencies in some elderly people.

3k-10. Pitch. The relation between frequency and the subjective magnitude of perceived pitch is shown by Table 3k-8. By definition, the pitch of a tone of 1,000 cps at 40 db SPL is 1,000 mels.[1]

TABLE 3k-8. PITCH OF A PURE TONE, IN MELS, AS A FUNCTION OF FREQUENCY

Frequency	Mels	Frequency	Mels	Frequency	Mels
20	0	350	460	1,750	1,428
30	24	400	508	2,000	1,545
40	46	500	602	2,500	1,771
60	87	600	690	3,000	1,962
80	126	700	775	3,500	2,116
100	161	800	854	4,000	2,250
150	237	900	929	5,000	2,478
200	301	1,000	1,000	6,000	2,657
250	358	1,250	1,154	7,000	2,800
300	409	1,500	1,296	10,000	3,075

3k-11. Localization of Sound. The localization of complex sounds is primarily a function of time differences of arrival at the two ears, and, to a first approximation, such differences may be calculated by assuming the ears on either end of the diameter of a sphere of 7.5 cm radius.

The localization of tones of low frequency (below 1,500 cps) is possible on the basis of phase differences, which may be interpreted in terms of time differences.

The localization of tones of high frequency is possible on the basis of intensity differences resulting from the sound shadow of the head. Exact measurements here are difficult at best.

Sound localization is greatly aided when the head or body can be rotated, or moved about, in the sound field, while the observer hears the appropriate sequence of sounds.[2]

Sound localization in reverberant rooms or with so-called "stereophonic-sound sources" depends critically upon a "precedence effect," by which the localization determined by the primary sound or sound from the nearer of two sound sources is overriding in its effect.[3]

In experiments where time differences are used to balance out intensity differences

[1] S. S. Stevens and J. Volkmann, The Relation of Pitch to Frequency: a Revised Scale, *Am. J. Psychol.* **53**, 329–353 (1940).

[2] H. Wallach, Ueber die Wahrnehmung der Schallrichtung, *Psychol. Forsch.* **22**, 238–266 (1938).

[3] H. Wallach, E. B. Newman, and M. R. Rosenzweig, The Precedence Effect in Sound Localization, *Am. J. Psychol.* **62**, 313–336 (1949).

in the opposite direction, 1.0×10^{-5} sec priority offsets a 6-db difference in intensity; 2.3×10^{-5} sec offsets a 14-db difference in intensity between the two ears.[1]

3k-12. Speech Power. The total radiated speech power, averaged over a 15-sec interval for a sample including both men and women at conversational levels used for telephone talking, has been estimated as 32 microwatts.

When measured at the face of a telephone transmitter, this power produces the sound pressure levels given in Table 3k-9 for different distances from the mouth of the speaker.[2]

TABLE 3k-9. AVERAGE SOUND PRESSURE LEVEL PRODUCED BY CONVERSATIONAL SPEECH AS A FUNCTION OF DISTANCE FROM LIPS TO MICROPHONE

	Distance, cm								
	Touching	0.5	1.0	2.5	5.0	10.0	25.0	50.0	100.0
Sound pressure level.......	104	102	99	95	90	85	78	72	66

A second source of variability lies in the essentially statistical distribution of speech power in time. If speech power is measured in successive $\frac{1}{8}$-sec intervals (a time slightly shorter than a syllable, and slightly longer than a phoneme), a distribution is obtained with the mean values given in Table 3k-9 and variability that can be attributed to time sampling equal to a standard deviation of 7.0 db.[3] The distribution is badly skewed so that the value 7.0 db indicates only a rough order of magnitude. The variability is also greater when particular frequency bands are measured.

A third source of variability is the variation in effort expended by the person who is talking. As a rough approximation, a raised voice level is 6 db above conversational level, the loudest level that can be maintained is 12 db above conversational level, and the loudest shout is 18 db above conversational level. In the other direction, a whisper may be 20 db below conversational level.

[1] J. H. Shaxby and F. H. Gage, Studies in the Localization of Sound. A. The Localization of Sounds in the Median Plane: An Experimental Investigation of the Physical Processes Concerned, *Med. Research Council (Brit.) Spec. Rept. Ser.* no. 166 (1932), 32 pp.

[2] M. H. Abrams, S. J. Goffard, J. Miller, F. H. Sanford, and S. S. Stevens, The Effect of Microphone Position on the Intelligibility of Speech in Noise, *OSRD Rept.* 4023 (1944), 16 pp.

[3] H. K. Dunn and S. D. White, Statistical Measurements on Conversational Speech, *J. Acoust. Soc. Am.* **11,** 278–288 (1940).

3k-13. Speech Sounds

TABLE 3k-10. CHARACTERISTICS OF SOUNDS IN GENERAL AMERICAN SPEECH

Symbol	Example	Power,* db re long time average†	Relative frequency of sound, %‡	Formant frequencies for men and women¶					
				First		Second		Third	
				M	W	M	W	M	W
u	cool	+0.6	1.60	300	370	870	950	2,240	2,670
ʊ	cook	+2.3	0.69	440	470	1,020	1,160	2,240	2,680
o	cone	+2.5	0.33	500	...	820			
ɔ	talk	+4.1	1.26	570	590	840	920	2,410	2,710
ɒ	cloth }	+3.7	{2.81	730	850	1,090	1,220	2,440	2,810
ɑ	calm }		0.49}						
a	ask }	+2.5	3.95	660	860	1,720	2,050	2,410	2,850
æ	bat }								
ɛ	bet	+1.6	3.44	530	610	1,840	2,330	2,480	2,990
e	tape	+1.4	1.84						
ɪ	bit	0.0	8.53	390	430	1,990	2,480	2,550	3,070
i	beet	0.0	2.12	270	310	2,290	2,790	3,010	3,310
ɚ	bird	−0.5	0.53	490	500	1,350	1,640	1,690	1,960
ə	sofa	.....	4.63						
ʌ	bun	+2.9	2.33	640	760	1,190	1,400	2,390	2,780
eɪ	laid	+1.4	see e						
aɪ	bite	+2.5	1.59						
ju	you	+0.6	0.31						
oʊ	soap	+2.5	1.30						
aʊ	about	+2.3	0.59						
ɔɪ	boil	+3.0	0.09						

* The power measurements do not represent the peak instantaneous power but the average over the sustained portion of the phoneme where such a period can be defined. In this case, as with the formant frequencies, the absolute values are highly variable, but intercomparisons among the various sounds are generally more reliable.

† H. Fletcher, "Speech and Hearing in Communication," p. 86, D. Van Nostrand Company, Inc., New York, 1953.

‡ G. Dewey, "Relative Frequency of English Speech Sounds," Harvard University Press, Cambridge, Mass., 1923.

¶ E. G. Richardson, ed., "Technical Aspects of Sound," pp. 215–217, Elsevier Press, Inc., New York, 1953.

TABLE 3k-10. CHARACTERISTICS OF SOUNDS IN GENERAL AMERICAN SPEECH (*Continued*)

Symbol	Example	Power,* db re long time average†	Relative frequency of sound, %‡	Formant frequencies for men and women¶			
				First	Second	Third	Fourth
l	lip	−3.0	3.74	450	1,000	2,550	2,950
m	me	−5.8	2.78	140	1,250	2,250	2,750
n	nip	−7.4	7.24	140	1,450	2,300	2,750
ŋ	sing	−4.4	0.96	140	2,350	2,750	
w	we	0.0	2.08				
r	rip	−1.0	6.35	500	1,350	1,850	3,500
j	yes	0.0	0.60	270	2,040		
p	pie	−15.2	2.04	. . .	800	1,350	
t	tie	−11.2	7.13	. . .	1,700	2,450	
k	key	−11.9	2.71	. . .	Variable		
b	by	−14.6	1.81	140	800	1,350	
d	die	−14.6	4.31	140	1,700	2,450	
g	guy	−11.2	0.74	140	Variable		
v	vie	−12.2	2.28	140	1,150	2,500	3,650
f	foe	−16.0	1.84	. . .	1,150	2,500	3,650
θ	thin	−23.0	0.37	. . .	1,450	2,550	
ð	then	−12.6	3.43	140	1,450	2,550	
s	sip	−11.0	4.55	. . .	2,000	2,700	
z	is	−11.0	2.97	140	2,000	2,700	
ʃ	shy	−4.0	0.82	. . .	2,150	2,650	
ʒ	measure	−10.0	0.05	140	2,150	2,650	
h	hit	−13.0	1.81				
tʃ	chop	−6.8	0.52				
dʒ	Joe	−9.4	0.44				

* The power measurements do not represent the peak instantaneous power but the average over the sustained portion of the phoneme where such a period can be defined. In this case, as with the formant frequencies, the absolute values are highly variable, but intercomparisons among the various sounds are generally more reliable.

† H. Fletcher, "Speech and Hearing in Communication," p. 86, D. Van Nostrand Company, Inc., New York, 1953.

‡ G. Dewey, "Relative Frequency of English Speech Sounds," Harvard University Press, Cambridge, Mass., 1923.

¶ E. G. Richardson, ed., "Technical Aspects of Sound," pp. 215–217, Elsevier Press, Inc., New York, 1953.

3k-14. Articulation Index. The articulation index is a set of numbers that makes possible the prediction of the efficiency of some types of voice-communication systems by the addition of suitably chosen values. The operations involve (1) dividing the speech spectrum into a series of bands having an equal possible contribution ΔA to the total efficiency, and (2) determining what proportion of the ΔA each band will contribute under the particular noise and speech conditions being tested.

Under (1) it is customary to use no more than 20 such bands. The frequency limits of 20 such bands are given in Table 3k-11.

TABLE 3k-11. TWENTY FREQUENCY BANDS CONTRIBUTING EQUALLY TO EFFICIENCY OF SPEECH COMMUNICATION*

Band No.	Frequency range	Band No.	Frequency range	Band No.	Frequency range
1	395	8	1,250–1,425	15	2,930–3,285
2	395–540	9	1,425–1,620	16	3,285–3,700
3	540–675	10	1,620–1,735	17	3,700–4,200
4	675–810	11	1,735–2,075	18	4,200–4,845
5	810–950	12	2,075–2,335	19	4,845–5,790
6	950–1,095	13	2,335–2,620	20	5,790
7	1,095–1,250	14	2,620–2,930		

* H. Fletcher, "Speech and Hearing in Communication," D. Van Nostrand Company, Inc., New York, 1953.

For conditions where substantial wide-band noise is present, the second requirement may be approximated by the formula

$$w_i = \tfrac{1}{30}(S_i - N_i + 6) \qquad \text{(3k-3)}$$

in which w_i is a weight having a maximum value of 1.0, S_i is the signal level in band i in decibels, N_i is the noise level in the same band i in decibels referred to the same base as S_i.[1]

TABLE 3k-12. ARTICULATION SCORES AS A FUNCTION OF ARTICULATION INDEX*

Articulation index	CVC syllables, %	Monosyllabic words (PB lists), %
0.10	7	7
0.20	22	22
0.30	38	40
0.40	55	61
0.50	68	77
0.60	79	87
0.70	87	93
0.80	93	96
0.90	96	98
1.00	98	99

* E. G. Richardson, ed., "Technical Aspects of Sound," Elsevier Press, Inc., New York, 1953.

The articulation index A is then described by the summation

$$A = \frac{1}{n}\sum_{i=1}^{i=n} w_i \qquad \text{(3k-4)}$$

Articulation scores are related to the articulation index according to the Table 3k-12.

[1] N. R. French and J. C. Steinberg, Factors Governing the Intelligibility of Speech Sounds, *J. Acoust. Soc. Am.* **19**, 90–119 (1947).

31. Classical Electro-dynamical Analogies

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Analogies are useful when it is desired to compare an unfamiliar system with one that is better known. The relations and actions are more easily visualized, the mathematics more readily applied, and the analytical solutions more readily obtained in the familiar system. Analogies make it possible to extend the line of reasoning into unexplored fields. In view of the tremendous amount of study which has been directed toward the solution of circuits, particularly electric circuits, and the engineer's familiarity with electric circuits, it is logical to apply this knowledge to the solutions of vibration problems in other fields by the same theory as that used in the solution of electric circuits. The objective in this section is the establishment of analogies between electrical, mechanical, and acoustical systems.

31-1. Resistance. *Electric Resistance.* Electric energy is changed into heat by the passage of an electric current through an electric resistance. Electric resistance R_E, in abohms, is defined as

$$R_E = \frac{e}{i} \tag{31-1}$$

where e = voltage across the electric resistance, abvolts
i = current through the electric resistance, abamp

Mechanical Rectilineal Resistance. Mechanical rectilineal energy is changed into heat by a rectilinear motion which is opposed by mechanical rectilineal resistance (friction). Mechanical rectilineal resistance (termed mechanical resistance when there is no ambiguity) R_M, in mechanical ohms, is defined as

$$R_M = \frac{f_M}{u} \tag{31-2}$$

where f_M = applied mechanical force, dynes
u = velocity at the point of application of the force, cm/sec

Mechanical Rotational Resistance. Mechanical rotational energy is changed into heat by a rotational motion which is opposed by a rotational resistance (rotational friction). Mechanical rotational resistance (termed rotational resistance when there is no ambiguity) R_R, in rotational ohms, is defined as

$$R_R = \frac{f_R}{\Omega} \tag{31-3}$$

where f_R = applied torque, dyne-cm
Ω = angular velocity about the axis at the point of the torque, radians/sec

Acoustic Resistance. Acoustic energy is changed into heat either by a motion in a fluid which is opposed by acoustic resistance due to a fluid resistance incurred by viscosity or by the radiation of sound. Acoustic resistance R_A, in acoustical ohms, is defined as

$$R_A = \frac{p}{U} \tag{31-4}$$

where p = pressure, dynes/sq cm
U = volume velocity, cu cm/sec

31-2. Inductance, Mass, Moment of Inertia, Inertance. *Inductance.* Electromagnetic energy is associated with inductance. Inductance is the electric-circuit element that opposes a change in current. Inductance L, in abhenrys, is defined as

$$e = L\frac{di}{dt} \tag{31-5}$$

where e = voltage, emf, or driving force, abvolts
$\frac{di}{dt}$ = rate of change of current, abamp/sec

Mass. Mechanical rectilineal inertial energy is associated with mass in the mechanical rectilineal system. Mass is the mechanical element which opposes a change in velocity. Mass m, in grams, is defined as

$$f_M = m\frac{du}{dt} \tag{31-6}$$

where $\frac{du}{dt}$ = acceleration, cm/sec/sec
f_M = driving force, dynes

Moment of Inertia. Mechanical rotational energy is associated with moment of inertia in the mechanical rotational system. Moment of inertia is the rotational element which opposes a change in angular velocity. Moment of inertia I, in gram (centimeter)^2, is defined as

$$f_R = I\frac{d\Omega}{dt} \tag{31-7}$$

where $\frac{d\Omega}{dt}$ = angular acceleration, radians/sec/sec
f_R = torque, dyne-cm

Inertance. Acoustic inertial energy is associated with inertance in the acoustic system. Inertance is the acoustic element which opposes a change in volume velocity. Inertance M, in grams per (centimeter)^4, is defined as

$$p = M\frac{dU}{dt} \tag{31-8}$$

where $\frac{dU}{dt}$ = rate of change of volume velocity, cu cm/sec/sec
p = driving pressure, dynes/sq cm

31-3. Electric Capacitance, Rectilineal Compliance, Rotational Compliance, Acoustic Capacitance. *Electric Capacitance.* Electric capacitance is associated with capacitance. Electric capacitance is the electric-circuit element which opposes a change in voltage. Electric capacitance C_E, in abfarads, is defined as

$$i = C_E\frac{de}{dt} \tag{31-9}$$

$$e = \frac{1}{C_E}\int i\,dt = \frac{Q}{C_E} \tag{31-10}$$

where Q = charge on the electrical capacitance, abcoulombs
e = emf, abvolts

Rectilineal Compliance. Mechanical rectilineal potential energy is associated with the compression of a spring or compliant element. Rectilineal compliance is the mechanical element which opposes a change in the applied force. Rectilineal compliance (termed compliance when there is no ambiguity) C_M, in centimeters per dyne, is defined as

$$f_M = \frac{x}{C_M} \tag{31-11}$$

where x = displacement, cm

f_M = applied force, dynes

Rotational Compliance. Mechanical rotational potential energy is associated with the twisting of a spring or compliant element. Rotational compliance is the mechanical element that opposes a change in the applied torque. Rotational compliance C_R, in radians per centimeter per dyne, is defined as

$$f_R = \frac{\phi}{C_R} \tag{31-12}$$

where ϕ = angular displacement, radians

f_R = applied torque, dyne-cm

Acoustic Capacitance. Acoustic potential energy is associated with the compression of a fluid or a gas. Acoustic capacitance is the acoustic element which opposes a change in the applied pressure. The acoustic capacitance C_A, in (centimeters)^5 per dyne, is defined as

$$p = \frac{X}{C_A} \tag{31-13}$$

where X = volume displacement, cu cm

p = pressure, dynes/sq cm

31-4. Representation of Electrical, Mechanical Rectilineal, Mechanical Rotational, and Acoustical Elements. Electrical, mechanical rectilineal, mechanical rotational,

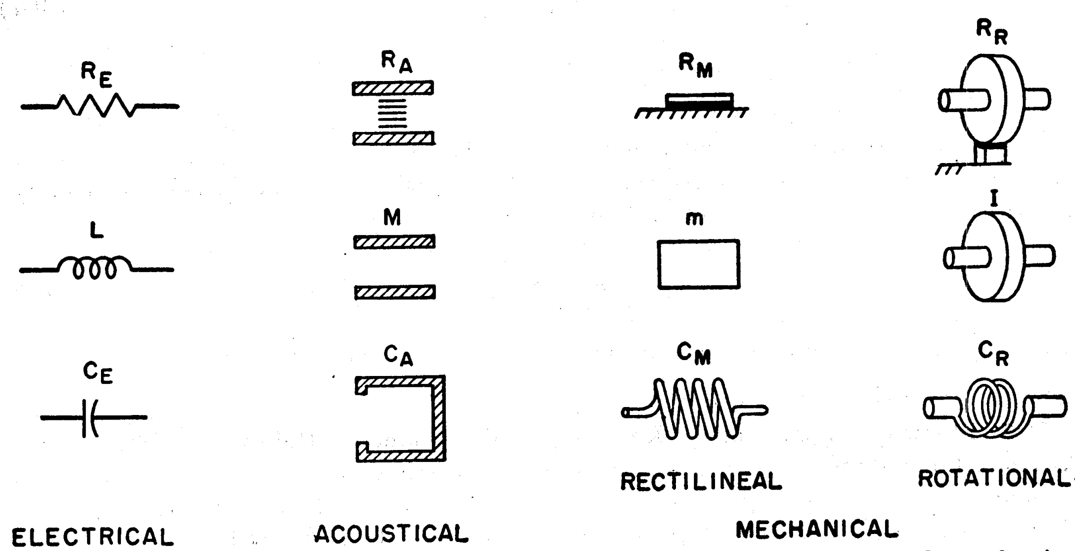


FIG. 31-1. Graphical representation of the three basic elements in electrical, mechanical rectilineal, mechanical rotational, and acoustical systems.

and acoustical elements have been defined in the preceding sections. Figure 31-1 illustrates schematically the three elements in each of the four systems.

The electrical elements, electric resistance, inductance, and electric capacitance are represented by the conventional symbols.

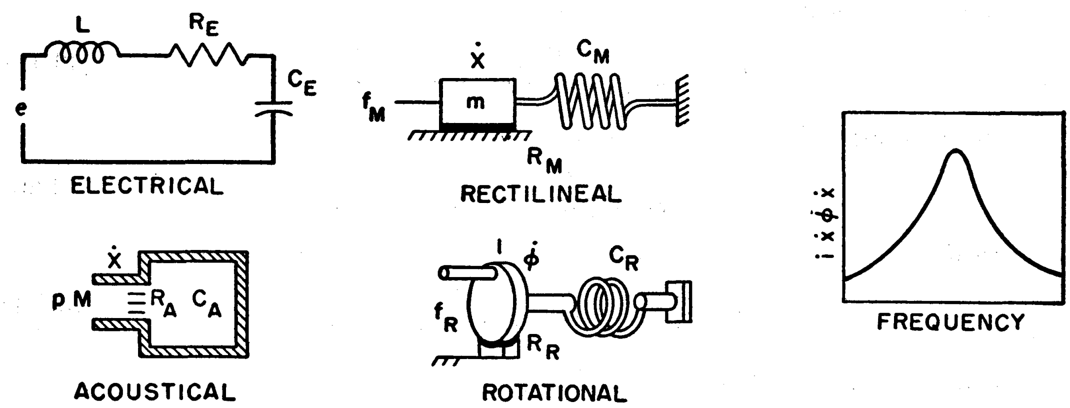
Mechanical rectilineal resistance is represented by sliding friction which causes dissipation. Mechanical rotational resistance is represented by a wheel with a sliding-friction brake which causes dissipation. Acoustic resistance is represented by narrow slits which causes dissipation due to viscosity when fluid is forced through the slits. These elements are analogous to electric resistance in the electrical system.

Inertia in the mechanical rectilineal system is represented by a mass. Moment of inertia in the mechanical rotational system is represented by a flywheel. Inertance in the acoustical system is represented as the fluid contained in a tube in which all the particles move with the same phase when actuated by a force due to pressure. These elements are analogous to inductance in the electrical system.

Compliance in the mechanical rectilineal system is represented as a spring. Rotational compliance in the mechanical rotational system is represented as a spring. Acoustic capacitance in the acoustical system is represented as a volume which acts as a stiffness or spring element. These elements are analogous to electric capacitance in the electrical system.

Table 31-1 shows the quantities, units, and symbols in the four systems.

31-5. Description of Systems of One Degree of Freedom. An electrical, mechanical rectilineal, mechanical rotational, and acoustical system of one degree of freedom are shown in Fig. 31-2. In one degree of freedom the activity in every element of the



MECHANICAL

FIG. 31-2. Electrical, mechanical rectilineal, mechanical rotation, and acoustical systems of one degree of freedom and the current, velocity, angular velocity and volume velocity response characteristics.

system may be expressed in terms of one variable. In the electrical system an electromotive force e acts upon an inductance L, an electric resistance R_E, and an electric capacitance C_E connected in series. In the mechanical rectilineal system a driving force f_M acts upon a particle of mass m fastened to a spring of compliance C_M and sliding upon a plate with a frictional force which is proportional to the velocity and designated as the mechanical rectilineal resistance R_M. In the mechanical rotational system a driving torque f_R acts upon a flywheel of moment of inertia I connected to a spring or rotational compliance C_R and the periphery of the wheel sliding against a brake with a frictional force which is proportional to the velocity and designated as the mechanical rotational resistance R_R. In the acoustical system, an impinging sound wave of pressure p acts upon an inertance M and an acoustic resistance R_A comprising the air in the tubular opening which is connected to the volume or acoustical capacitance C_A. The acoustic resistance R_A is due to viscosity.

The differential equations describing the four systems of Fig. 31-2 are as follows:
Electrical

$$L\ddot{q} + R_E\dot{q} + \frac{Q}{C_E} = E\epsilon^{j\omega t} \qquad (31\text{-}14)$$

Mechanical rectilineal

$$m\ddot{x} + R_M\dot{x} + \frac{x}{C_M} = F_M\epsilon^{j\omega t} \tag{31-15}$$

Mechanical rotational

$$I\ddot{\phi} + R_R\dot{\phi} + \frac{\phi}{C_R} = F_R\epsilon^{j\omega t} \tag{31-16}$$

Acoustical

$$M\ddot{X} + R_A\dot{X} + \frac{X}{C_A} = P\epsilon^{j\omega t} \tag{31-17}$$

E, F_M, F_R, and P are the amplitudes of the driving forces in the four systems. $E\epsilon^{j\omega t} = e$, $F_M\epsilon^{j\omega t} = f_M$, $F_R\epsilon^{j\omega t} = f_R$ and $P\epsilon^{j\omega t} = p$.

The steady-state solutions of Eqs. (31-14) to (31-17) are:

Electrical

$$\dot{q} = i = \frac{E\epsilon^{j\omega t}}{R_E + j\omega L - (j/\omega C_E)} = \frac{e}{Z_E} \tag{31-18}$$

Mechanical rectilineal

$$\dot{x} = \frac{F\epsilon^{j\omega t}}{R_M + j\omega m - (j/\omega C_M)} = \frac{f_M}{Z_M} \tag{31-19}$$

Mechanical rotational

$$\dot{\phi} = \frac{F\epsilon^{j\omega t}}{R_R + j\omega I - (j/\omega C_R)} = \frac{f_R}{Z_R} \tag{31-20}$$

Acoustical

$$\dot{X} = \frac{P\epsilon^{j\omega t}}{R_A + j\omega M - (j/\omega C_A)} = \frac{p}{Z_A} \tag{31-21}$$

The vector electric impedance is

$$Z_E = R_E + j\omega L - \frac{j}{\omega C_E} \tag{31-22}$$

The vector mechanical rectilineal impedance is

$$Z_M = R_M + j\omega m - \frac{j}{\omega C_M} \tag{31-23}$$

The vector mechanical rotational impedance is

$$Z_R = R_R + j\omega I - \frac{j}{\omega C_R} \tag{31-24}$$

The vector acoustic impedance is

$$Z_A = R_A + j\omega M - \frac{j}{\omega C_A} \tag{31-25}$$

TABLE 31-1. QUANTITIES, UNITS, AND SYMBOLS FOR ELECTRICAL, MECHANICAL RECTILINEAL, MECHANICAL ROTATIONAL, AND ACOUSTICAL ELEMENTS

Electrical			Mechanical rectilineal		
Quantiy	Unit	Symbol	Quantity	Unit	Symbol
Electromotive force	Volts × 10^8	e	Force	Dynes	f_M
Charge or quantity	Coulombs × 10^{-1}	Q	Linear displacement	Centimeters	x
Current	Amperes × 10^{-1}	i	Linear velocity	Centimeters per second	$\dot{x}$ or u
Electric impedance	Ohms × 10^9	Z_E	Mechanical impedance	Mechanical ohms	Z_M
Electric resistance	Ohms × 10^9	R_E	Mechanical resistance	Mechanical ohms	R_M
Electric reactance	Ohms × 10^9	X_E	Mechanical reactance	Mechanical ohms	X_M
Inductance	Henry × 10^9	L	Mass	Grams	m
Electric capacitance	Farads × 10^{-9}	C_E	Compliance	Centimeters per dyne	C_M
Power	Ergs per second	P_E	Power	Ergs per second	P_M

Mechanical rotational			Acoustical		
Quantity	Unit	Symbol	Quantity	Unit	Symbol
Torque	Dyne-centimeter	f_R	Pressure	Dynes per square centimeter	p
Angular displacement	Radians	ϕ	Volume displacement	Cubic centimeters	X
Angular velocity	Radians per second	$\dot{\phi}$ or Ω	Volume velocity	Cubic centimeters per second	$\dot{X}$ or U
Rotational impedance	Rotational ohms	Z_R	Acoustic impedance	Acoustic ohms	Z_A
Rotational resistance	Rotational ohms	R_R	Acoustic resistance	Acoustic ohms	R_A
Rotational reactance	Rotational ohms	X_R	Acoustic reactance	Acoustic ohms	X_A
Moment of inertia	(Gram) (centimeter)^2	I	Inertance	Grams per (centimeter)^4	M
Rotational compliance	Radians per dyne per centimeter	C_R	Acoustic capacitance	(Centimeter)^5 per dyne	C_A
Power	Ergs per second	P_R	Power	Ergs per second	P_A

3m. The Mobility and Classical Impedance Analogies[1]

FLOYD A. FIRESTONE

Editor, The Journal of the Acoustical Society of America

3m-1. Introduction. An *analogy* is a recognized relationship of consistent mutual similarity between the equations and structures appearing within two or more fields of knowledge, and an identification and association of the quantities and structural elements which play mutually similar roles in these equations and structures, for the purpose of facilitating transfer of knowledge of mathematical procedures of analysis and behavior of the structures between these fields.

The theory of analogies is still developing, as evidenced by the recent publications of Olson, Raymond, Bloch, Trent, Le Corbeiller, Bauer, Beranek, and others (see references on page 3-177). This section sets forth the author's recommendations for a useful problem-solving technique as presented in his paper, 'Twixt Earth and Sky with Rod and Tube, *J. Acoust. Soc. Am.* **26,** 140 (1954) (abstract only).

Instead of drawing an analogous electric circuit, the author recommends that *mechanical* and *acoustical* schematic diagrams be drawn, utilizing the mechanical and acoustical symbols shown below. Such a schematic diagram can be drawn directly from an inspection of the structure and is a record of our determinations of the functions and connections of its parts. The mechanical and acoustical symbols here presented are distinctive but similar to their electrical analogues so that their shape indicates to one familiar with electrical schematics the algebraic operations which are to be performed in the analysis. Then the problem is solved using mechanical or acoustical units. Even if an analogous electric circuit is drawn, there is advantage in understanding in detail the mechanical or acoustical analogue of each straight line, junction, and element on the diagram, as set forth in the mobility and impedance analogy tables which follow.

Schematic diagrams based on analogies are most useful in solving those mechanical and acoustical problems where it is known at the outset that the parts are constrained to move in one line only. Problems involving several degrees of freedom for each mass require the construction of a separate schematic diagram for each degree of freedom, usually with coupling between these diagrams.

In the *mobility analogy,* mechanical mobility (complex velocity amplitude divided by complex force amplitude) is analogous to electric impedance, velocity to voltage, and force to current. In the *impedance analogy,* mechanical *impedance* is analogous to electric impedance, force to voltage, and velocity to current.

3m-2. Wires, Rods, and Tubes. In an electric circuit, connections between distant terminals are made by slim wires which, when idealized on a schematic diagram, are assumed to be free of inductance, resistance, and capacitance to ground. Ofttimes a number of wires are soldered together to form a (soldered) *junction* which

[1] The author wishes to acknowledge with pleasure many interesting and instructive conversations on this subject, as well as a voluminous correspondence, with Dr. Horace M. Trent.

ensures equal voltages at all the terminals connected by the tree of wires. But it will also be useful to introduce the *isocurrent junction* (or electric mesher) which ensures equal currents in all the wires coming to it; structurally it is a set of similar ideal transformers with one side of each primary grounded and all secondaries connected in series, the schematic symbol being abbreviated to that of a junction with a circle around it indicating the series of secondaries. The isocurrent junction is the electric example of that broad class of junctions which we shall call "meshers" because they have the effect of connecting all the attached circuits into the same mesh.

In a mechanical system, on the other hand, the connections between distant moving terminals are in practice made by either or both of *two* "slim" devices, *rods* or *tubes*. Ideally, the rods are free from mass, friction, or compliance. Ideally the hydraulic tubes are held stationary and are filled with ideal fluid free from mass, viscosity, or compressibility. Ofttimes a number of rods are bolted together to form a *rigid junction* which ensures equal velocities of all the terminals connected by the tree of rods. Also, ofttimes a number of tubes are joined in a small common chamber to form a *hydraulic junction* which ensures equal pressures (and forces if all tubes are of the same area) at all the terminals connected by the tree of tubes. However, rods can be joined in a hydraulic junction which will ensure equal forces in the rods, if the rods are provided with equal-area pistons hydraulically connected. Similarly, by means of connected pistons, tubes can be joined in a rigid junction.

Since mechanical systems are customarily connected by two kinds of slim devices (rods and tubes) while electric systems are connected by only one kind of slim device (wires), it is not possible in general to draw a correct schematic diagram by either the mobility or impedance analogies *alone* which will correspond completely to the apparent geometry of the mechanical structure. A mobility schematic is a rod diagram (each straight line represents a rod), and it will correspond with the geometry of all parts of the mechanical structure which are rigidly connected by rods. An impedance schematic is a tubing diagram (each connecting line represents a hydraulic tube), and it will correspond with the geometry of all parts of the mechanical structure which are hydraulically connected by tubes.

3m-3. Ground, Earth, and Sky. In a mobility schematic or rod diagram, the reference symbol which is analogous to the ground symbol in an electrical wiring diagram is a frame of reference called the *earth*, whereas in an impedance schematic or tubing diagram the reference symbol is a force (or pressure) of reference called the *sky*. The sky is the dual of the earth. Structurally, the sky consists of a bowl, a lake, or an atmosphere of ideal fluid maintained under a constant pressure of reference. In a mobility schematic or rod diagram one terminal of every mass is the earth relative to which the velocity of the mass is measured, while in an impedance schematic or tubing diagram one terminal of every spring is the sky relative to which the force in the spring is measured; these concepts are necessary in order that either type of schematic diagram may be drawn by inspection. The earth has zero mobility and infinite mass, while the sky has zero impedance and infinite compliance.

3m-4. Analogues of the Condenser and the Capacitor. In addition to the mechanical analogues of the inductor and the resistor, the analogues of *two* classes of capacitive elements must be considered, which we shall distinguish by the names *condenser* and *capacitor*. The condenser is the parallel-plate device which can be connected either in a high wire or to ground, while the capacitor is typified by the isolated sphere in free space as discussed in electrostatics, one terminal only being free while the other terminal is permanently grounded.

In the mobility analogy, every mass is analogous to the capacitor, not the condenser, in the sense that one terminal of the mass is the body of the mass while the other terminal is always the earth relative to which the velocity of the mass is measured. It is this drawing of an earth symbol near each mass which makes closed circuits in a mobility schematic and permits the drawing of a correct rod diagram of any rod-connected system in a straightforward intuitive manner. There is also an unusual structure called a *transinertor* which is a combination of two masses and a mesher, and which is analogous to the condenser; it can be connected either in series with the high rod or to ground.

In the impedance analogy, every spring is analogous to the capacitor, not the condenser, in the sense that one terminal of the spring is the body of the spring while the other terminal is always the sky relative to which the force of the spring is measured. It is this drawing of a sky symbol near each spring which makes closed circuits in an impedance schematic and permits the drawing of a correct tubing diagram of any tube-connected system in a straightforward intuitive manner.

3m-5. The Dotted Arrow. Alongside each rod diagram is drawn a dotted arrow, usually toward the right, which indicates the direction of motion which is considered positive. This is analogous to marking the plus and minus signs on our voltmeters. A solid arrow superimposed on a rod indicates the direction in which impulse is flowing, such as would increase the momentum of a mass in the direction of the dotted arrow. If the solid arrow is in the direction of the dotted arrow, the rod is in compression; if the arrows are in opposite directions, the rod is in tension; if the arrows are at right angles, the rod is in shear. In a rotational system, the rotational velocity is considered positive if it is clockwise when looking in the direction of the dotted arrow placed beside the rotational schematic diagram; a solid arrow superimposed on a shaft then indicates the direction of flow of torsional impulse such as would increase the positive angular momentum of any inertor into which it flows.

In a tubing diagram only the solid arrow is used, superimposed on a tube. It indicates the direction of positive fluid velocity or volume velocity.

3m-6. Rationale of the Schematic Symbols Proposed for the Elements. In both analogies, the mechanical and acoustical schematic symbols are similar in appearance to their analogous electrical symbols. On a rod diagram, the symbols for a mechanical spring and responsor have $1\frac{1}{2}$ "wiggles," the acoustic elastor and responsor have $2\frac{1}{2}$ wiggles, the torsional spring and responsor have $2\frac{1}{2}$ wiggles but are tapered, while the electrical inductor and resistor have $3\frac{1}{2}$ wiggles as usual. Similarly a torsional inertor is tapered.

On an impedance schematic, the symbols are similar to those for the mobility schematic though dual in meaning, and each impedance schematic symbol has a line drawn beside it.

3m-7. Method of Drawing Schematic Diagrams. (1) Choose your analogy, either for life or for the problem at hand, remembering that the mobility analogy is the most convenient for rod-connected systems while the impedance analogy is the most convenient for hydraulic tube-connected systems. (2) Identify the functions performed by each part of the given structure. (3) Choose the schematic symbols which represent these functions. (4) Identify the terminals of each element, coupler, and vibrator of the structure. (5) Connect in the schematic diagram by means of appropriate connectors and rigid or hydraulic junctions those terminals which are connected in the structure.

The identification of the terminals of each element will include the assignment of the earth symbol as one terminal of each mass in a mobility schematic or rod diagram, or the assignment of the sky symbol as one terminal of each spring in an impedance schematic or tubing diagram; this will result in closed meshes and correct series and parallel connections in each diagram.

A single hydraulic tube fitted with pistons at its two ends (of equal areas for mechanical systems but not necessarily equal for acoustic systems) performs the same functions as a rod; so both rod and tube may be represented by a straight line and are interchangeable in a series. It is where several rods join, or tubes join, that there is a difference of function, a *rigid junction* of rods ensuring equal velocities while a *hydraulic junction* of tubes ensures equal forces or sound pressures. Either analogy may therefore be used for diagraming a system connected by rods and/or tubes by first determining whether a given structural connection performs the function of a rigid junction or of a hydraulic junction, then designating by means of the symbols below whether the connection constitutes a *simple junction* (analogous to a soldered junction of wires) or a *mesher* (analogous to the isocurrent junction mentioned in Sec. 3m-2). That analogy will be best for a given problem which brings in a minimum number of meshers with which we are not so familiar. Thus the mobility analogy will be best for rod-connected systems and the impedance analogy best for tube-connected systems.

3m-8. Types of Schematic Diagram. 1. The mobility schematic diagram or rod diagram. Because most mechanical systems are rod-connected and have no tubes, a mobility schematic diagram or rod diagram will be most convenient and can usually be drawn by inspection of the structure, using the mobility-analogy symbols given on left pages. Even an acoustic system of the kind where there are no side branches and the elements are of equal cross-sectional areas and lie in a series, as when a piezoelectric crystal radiates plane waves into a delay line, may be most conveniently represented by a mobility schematic since the contact of the adjacent faces of the elements ensures their equal volume velocities as if they were connected by acoustic rods.

2. The impedance schematic diagram or tubing diagram. If we have a hydraulically operated mechanical system which is tube-connected or an acoustic filter connected by tubes with side branches, an impedance schematic or tubing diagram will be most convenient and can be drawn by inspection using the impedance-analogy symbols given on right pages, provided that the sky is introduced as one terminal of each spring.

3. The two-analogy schematic diagram. Complete correspondence between the schematic diagram and the geometry of the structure can be obtained by diagraming the rod-connected parts by the mobility analogy and the tube-connected parts by the impedance analogy, appropriate couplers being indicated where rod and tube portions adjoin. Using this technique, the schematic diagram of the system can be drawn by inspection of the original structure, using the appropriate mechanical or acoustical symbols given below, including the analogy connectors on page 3-176.

3m-9. Mechanical Mobility z vs. Mechanical Admittance Y_M. Why should the new term *mechanical mobility* z be introduced when it is of the same *magnitude* as the established term *mechanical admittance* Y_M?

$$\text{Mechanical mobility } z = \frac{\hat{v} \text{ across}}{\hat{F} \text{ through}} \text{ while mechanical admittance } Y_M = \frac{\hat{v} \text{ through}}{\hat{F} \text{ across}}.$$

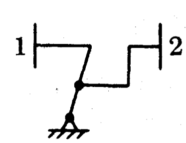
Thus while the magnitudes of the mechanical mobility and mechanical admittance are equal, the words *through* and *across* are inverted in the definitions, because mobility belongs in a rod-connected system and mechanical admittance belongs in a tube-connected system.

Mechanical mobility z is indigenous to a rod-connected system, and when a number of springs or other elements are connected in series, the mobility of the combination is the sum of the individual mobilities: $z = z_1 + z_2 + z_3$. It would be *unanalogous*, though correct, to say that the mechanical admittance of the elements in *series* is the sum of the individual mechanical admittances; this lack of analogy is avoided by introducing with the rod diagram the new term *mobility* and having it associated with the letter z. Mechanical admittance is indigenous to a tube-connected system and the above-mentioned *series* of structural elements would turn out to be a *parallel* combination of elements in a *tubing* diagram; the mechanical admittance of the

[*Text continued on page 3-177; tables, over.*]

THE MOBILITY ANALOGY

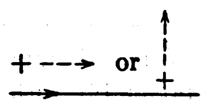
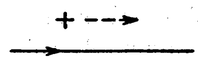
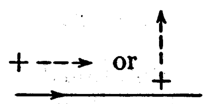
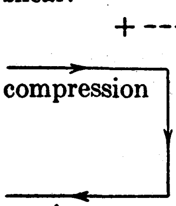
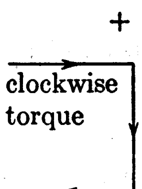
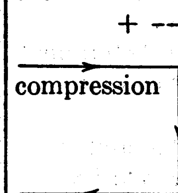
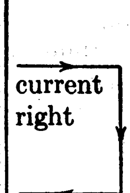
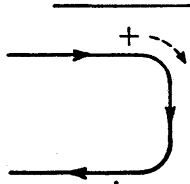
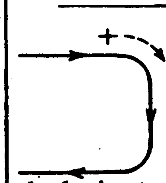
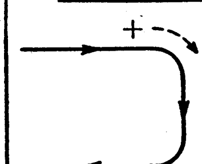
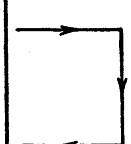
Symbols for Constructing Mechanical, Acoustical, and Electrical Schematic Diagrams Based on the Mobility Analogy

Symbols for rod diagrams			Symbols for the analogous wiring diagrams
Rectilineal mechanical systems	Rotational mechanical systems	Acoustic systems (preferably rigidly connected)	Electric circuits (preferably with soldered junctions)
Connectors			
1 ——— 2 rod	1 ——— 2 shaft	1 ——— 2 acoustic rod	1 ——— 2 wire
Ideal massless, incompressible, frictionless rod, *not* necessarily of uniform cross section, which connects movable terminals 1 and 2 so that	Ideal inertialess, uncompliant, frictionless shaft, *not* necessarily of uniform cross section, which connects movable terminals 1 and 2 so that	Ideal inertanceless, uncompliant, frictionless acoustic rod which connects two wavefronts 1 and 2, *not* necessarily of the same areas, by any means which function the same as a pivoted lever driving pistons 1 and 2 at lever arms equal to the reciprocal of the piston areas, so that	Ideal capacitanceless, inductanceless, resistanceless wire, *not* necessarily of uniform cross section, which connects terminals 1 and 2 so that
velocity $v_1 = v_2$ and force $F_1 = F_2$ Propagation in any type of ideal rod is considered instantaneous	angular velocity $v_{R1} = v_{R2}$ and torque $F_{R1} = F_{R2}$	the volume velocity $U_1 = U_2$ and the sound pressure $p_1 = p_2$ Frequent special case, equal areas in contact.  $l_1A_1 = l_2A_2 = 1$ Prototypal structure of acoustic rod	voltage $E_1 = E_2$ and current $I_1 = I_2$ Propagation in an ideal wire is considered instantaneous

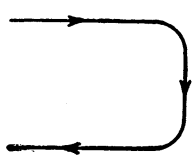
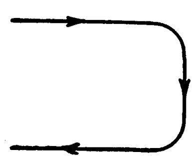
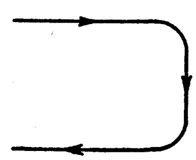
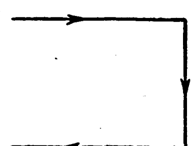
THE IMPEDANCE ANALOGY (Classical Analogy)
Symbols for Constructing Mechanical, Acoustical, and Electrical Schematic Diagrams Based on the Impedance Analogy

Symbols for tubing diagrams			Symbols for the analogous wiring diagrams
Rectilineal mechanical systems	Rotational mechanical systems	Acoustic systems (preferably tube-connected)	Electric circuits (preferably with soldered junctions)
Connectors			
1 ——— 2 tube	1 ——— 2 tube	1 ——— 2 acoustic tube	1 ——— 2 wire
Stationary tube of *unit area* of cross section, filled with ideal, massless, incompressible, inviscid fluid, often terminating in unit-area pistons; or any mechanism of equivalent function. Dead-end tubes are closed unless connected to *sky*.	Stationary tube filled with ideal fluid, and having at every end *identical* fluid motors for transducing lineal fluid motion to rotation of a solid or fluid member (example, the Sperry Exactor hydraulic control); or any mechanism of equivalent function.	Stationary tube, *not* necessarily of uniform cross section, filled with ideal fluid (tube diameter being small compared with the wavelength in the actual medium). The speed of sound in an ideal acoustic tube is infinite.	Stationary ideal, capacitanceless, inductanceless, resistanceless wire, *not* necessarily of uniform cross section. Propagation along an ideal wire is considered instantaneous.
It connects terminals 1 and 2 so that force $F_1 = F_2$ and velocity $v_1 = v_2$	It connects terminals 1 and 2 so that torque $F_{R1} = F_{R2}$ and angular velocity $v_{R1} = v_{R2}$	It connects terminals 1 and 2 so that sound pressure $p_1 = p_2$ and volume velocity $U_1 = U_2$	It connects terminals 1 and 2 so that voltage $E_1 = E_2$ and current $I_1 = I_2$

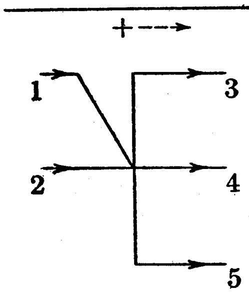
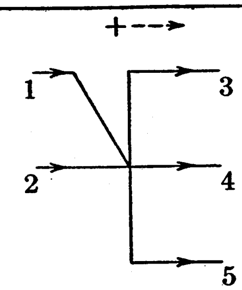
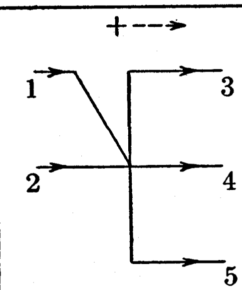
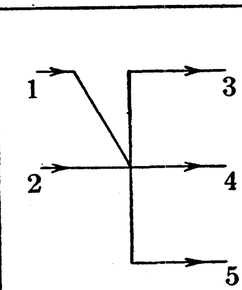
THE MOBILITY ANALOGY

Rectilineal mechanical	Rotational mechanical	Acoustic	Electric
	Symbols Indicating the Signs of the Variables		
 Dotted arrow shows direction of positive velocity v relative to the earth.	 Clockwise rotation v_R looking in the direction of the dotted arrow is positive relative to the earth.	 Dotted arrow shows direction of positive volume velocity U relative to the earth.	 The + sign near the wire indicates a positive voltage relative to ground.
Small arrow on rod shows direction of positive flow of force F and impulse Q_M. If impulse flows into a mass, its momentum in the direction of the dotted arrow is increased. Both arrows in the same direction indicates compression; at right angles, shear.	Small arrow on shaft shows direction of positive flow of torque F_R and torsional impulse Q_R.	Small arrow shows direction of flow of force per unit area (sound pressure p) and impulse per unit area (acoustic impulse Q_A). Both arrows in the same direction indicates compression; at right angles, shear.	Arrow shows direction of flow of positive current I and charge Q.
 compression; shear; tension Rigid connector offset	 clockwise torque; (chain, belt, or gears, 1 to 1); counter-clockwise torque Rigid connector offset	 compression; shear; rarefaction Rigid connector offset (rare)	 current right; current downward; current left Offset connector
 compression everywhere Flexible connector	 clockwise torque everywhere Flexible shaft	 compression everywhere Flexible connector (similar to tube of small d/λ)	 same as above Offset connector

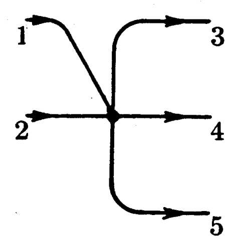
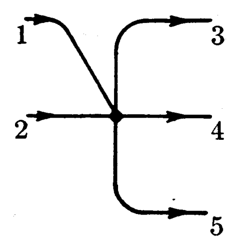
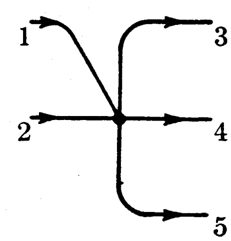
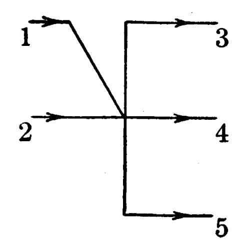
THE IMPEDANCE ANALOGY

Rectilineal mechanical	Rotational mechanical	Acoustic	Electric
Symbols Indicating the Signs of the Variables			
+ →	+ →	+ →	+ →
The + sign indicates positive pressure in the tube (relative to the sky) and is assumed to be associated with positive force F	The + sign indicates positive pressure in the tube (relative to the sky) and is assumed to be associated with positive torque F_R	The + sign indicates a positive sound pressure p (relative to the sky, P_0)	The + sign indicates a positive voltage (relative to the ground)
The arrow shows the direction of the velocity v (relative to the tubing)	The arrow shows the direction of fluid motion which is associated with positive clockwise angular velocity v_R	The arrow shows the direction of a positive volume velocity U	The arrow shows the direction of flow of a positive historical current I
Rounded corners are recommended for tubes			
 v	 v_R	 U	 I
Stationary tube with bends; direction of positive velocity v and displacement s is shown by arrows	Stationary tube with changes of direction; positive clockwise angular velocity v_R and angular displacement s_R are shown by arrows	Stationary tube with bends; direction of positive volume velocity U and volume displacement S_A is shown by arrows	Offset connector; wire with bends; direction of positive current I and flow of charge Q is shown by arrows

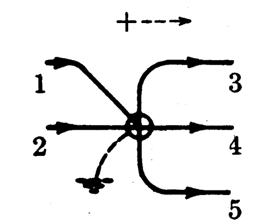
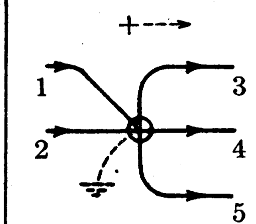
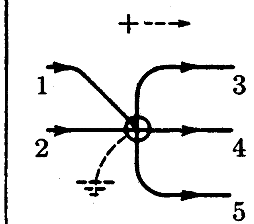
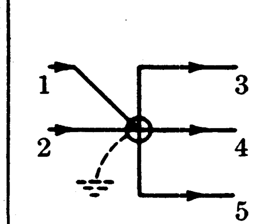
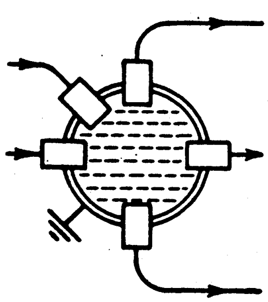
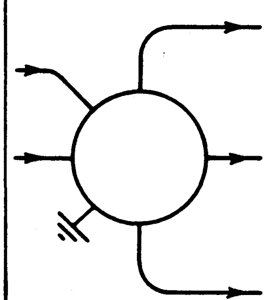
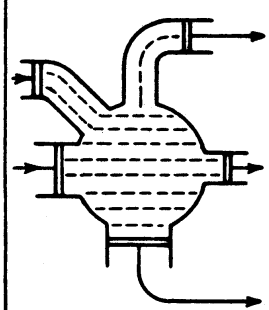
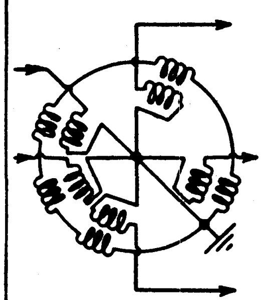
THE MOBILITY ANALOGY

Rectilineal mechanical	Rotational mechanical	Acoustic	Electric
Simple junctions. The *across* variables are equal. The sum of the *through* variables toward the junction is zero			
			
Rigid junction (welded rods of any cross-sectional area)	Rigid junction (1 to 1 gearbox with shafts of any cross-sectional area)	Rigid acoustic junction, connecting wavefronts 1 to 5, *not* necessarily of equal areas, by any means which functions the same as a pivoted lever driving pistons 1 to 5 at lever arms equal to the reciprocals of the piston areas (as if the levers of individual acoustic rods were rigidly connected) $1 = l_1A_1 = l_2A_2 = \text{etc.}$	Junction (soldered wires of any cross-sectional area)
$v_1 = v_2 = \text{etc.}$ $F_1 + F_2 - F_3 - F_4 - F_5 = 0$	$v_{R1} = v_{R2} = \text{etc.}$ $F_{R1} + F_{R2} - F_{R3} - F_{R4} - F_{R5} = 0$	$U_1 = U_2 = \text{etc.}$ $p_1 + p_2 - p_3 - p_4 - p_5 = 0$	$E_1 = E_2 = \text{etc.}$ $I_1 + I_2 - I_3 - I_4 - I_5 = 0$

THE IMPEDANCE ANALOGY

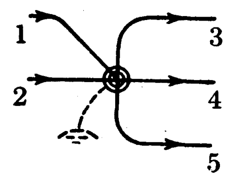
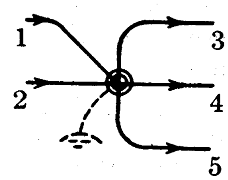
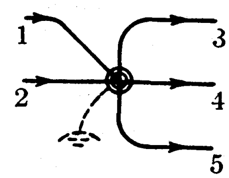
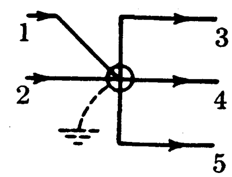
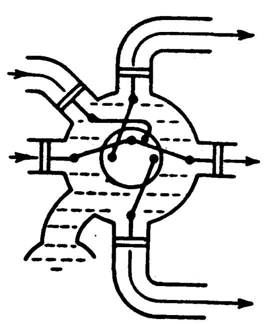
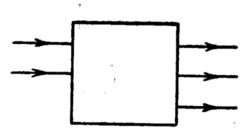
Rectilineal mechanical	Rotational mechanical	Acoustic	Electric
Simple junctions. The *across* variables are equal. The sum of the *through* variables toward the junction is zero			
 Hydraulic junction, having unit terminal areas 1 to 5	 Hydraulic junction; (differential gear-box 1:1)	 Hydraulic acoustic junction of tubes *not* necessarily of equal areas	 Junction (soldered wires of any cross-sectional area)
$F_1 = F_2 =$ etc. $v_1 + v_2 - v_3 - v_4 - v_5 = 0$	$F_{R1} = F_{R2} =$ etc. $v_{R1} + v_{R2} - v_{R3} - v_{R4} - v_{R5} = 0$	$p_1 = p_2 =$ etc. $U_1 + U_2 - U_3 - U_4 - U_5 = 0$	$E_1 = E_2 =$ etc. $I_1 + I_2 - I_3 - I_4 - I_5 = 0$

THE MOBILITY ANALOGY

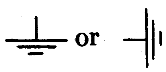
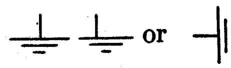
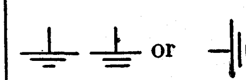
Rectilinear mechanical	Rotational mechanical	Acoustic	Electric
Meshers. The *through* variables (in the direction of the dotted arrow) are equal. The sum of the *across* variables (with sign changed where through arrow points away from junction) is zero			
			
Hydraulic junction; mesher (see below) $F_1 = F_2 =$ etc. $v_1 + v_2 - v_3 - v_4 - v_5 = 0$ The earth connection is not necessary when equal numbers of forces flow *to* and *from* the mesher. Typical hydraulic junction structure (see symbol above):	Hydraulic junction; mesher (see below) $F_{R1} = F_{R2} =$ etc. $v_{R1} + v_{R2} - v_{R3} - v_{R4} - v_{R5} = 0$ The earth connection is not necessary when equal numbers of torques flow *to* and *from* the mesher. Typical hydraulic junction structure (see symbol above):	Hydraulic acoustic junction; mesher $p_1 = p_2 =$ etc. $U_1 + U_2 - U_3 - U_4 - U_5 = 0$ The acoustic earth connection is not necessary when equal numbers of sound pressures flow *to* and *from* the mesher. Typical hydraulic junction structure (see symbol above):	Isocurrent junction; mesher (see below) $I_1 = I_2 =$ etc. $E_1 + E_2 - E_3 - E_4 - E_5 = 0$ The ground connection is not necessary when equal numbers of currents flow *to* and *from* the mesher. Typical mesher structure (see symbol above):
			
Equal-area pistons with common liquid, or any set of levers which will ensure equal forces of compression	Differential gearbox giving equal torques	Acoustic rods, generally of different wavefront (piston) areas, entering a small chamber containing ideal fluid	Similar ideal transformers (on separate cores) with secondaries in series. Phased as shown by arrows

THE IMPEDANCE ANALOGY

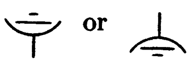
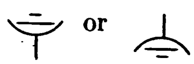
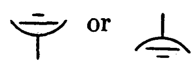
Meshers. The *through* variables (as indicated by the arrows) are equal. The sum of the *across* variables (with sign changed where through variable arrow points away from junction) is zero

Rectilinear mechanical	Rotational mechanical	Acoustic	Electric
			
Rigid junction; mesher (see below) $v_1 = v_2 =$ etc. $F_1 + F_2 - F_3 - F_4 - F_5 = 0$ The sky connection is not necessary when equal numbers of velocities flow *to* and *from* the mesher. Typical rigid junction structure (see symbol above):	Rigid (geared) junction; mesher $v_{R1} = v_{R2} =$ etc. $F_{R1} + F_{R2} - F_{R3} - F_{R4} - F_{R5} = 0$ The sky connection is not necessary when equal numbers of rotational velocities flow *to* and *from* the mesher. Typical rigid junction structure (see symbol above):	Rigid acoustic junction; mesher $U_1 = U_2 =$ etc. $p_1 + p_2 - p_3 - p_4 - p_5 = 0$ The acoustic sky connection is not necessary when equal numbers of volume velocities flow *to* and *from* the mesher. Typical rigid acoustic junction structure:	Isocurrent junction; mesher $I_1 = I_2 =$ etc. $E_1 + E_2 - E_3 - E_4 - E_5 = 0$ The ground connection is not necessary when equal numbers of currents flow *to* and *from* the mesher. Typical mesher structure:
 A multiplicity of rigidly connected equal-area pistons which ensure equal fluid velocities in all tubes; or a number of rods bolted together	 gearbox Gears, belts, chains, or levers, which ensure equal rotations of shafts	Same as the device in the rectilineal column at the left, but the pivoted wheel drives pistons 1 to 5, not necessarily of equal areas, at lever arms equal to the reciprocals of the areas. Special case, equal-area pistons rigidly connected; or wavefronts in contact	Same as on opposite page

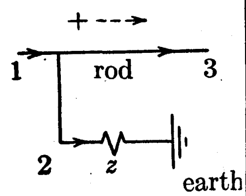
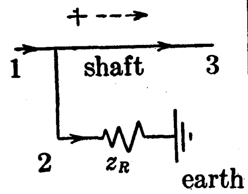
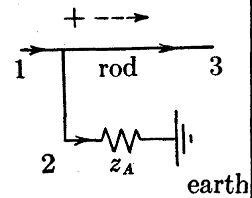
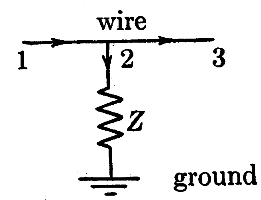
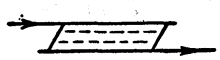
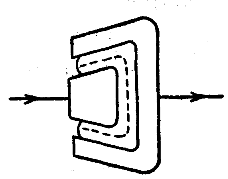
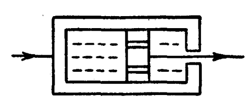
THE MOBILITY ANALOGY

Rectilineal mechanical	Rotational mechanical	Acoustic	Electric
Referents			
 Earth, velocity of reference, frame of reference	 Earth, angular velocity of reference, frame of reference	 Acoustic earth, volume velocity of reference, frame of reference	 Ground, voltage of reference
$v = 0$	$v_R = 0$	$U = 0$	$E = 0$
Force F is measured relative to the force in the surrounding empty space (no symbol)	Torque F_R is measured relative to the torque in the surrounding empty space (no symbol)	Sound pressure p is measured relative to the pressure P_0 in the surrounding atmosphere (no symbol)	Current I is measured relative to the current in the surrounding empty space (no symbol)
Passive Elements			
 z	 z_R	 z_A	 Z
A mechanical mobility z.	A rotational mobility z_R.	An acoustic mobility z_A.	An electric impedance Z.
$z \equiv \dfrac{\hat{v} \text{ across}}{\hat{F} \text{ through}}$	$z_R \equiv \dfrac{\hat{v}_R \text{ across}}{\hat{F}_R \text{ through}}$	$z_A \equiv \dfrac{\hat{U} \text{ across}}{\hat{p} \text{ through}}$	$Z \equiv \dfrac{\hat{E} \text{ across}}{\hat{I} \text{ through}}$
$\hat{v}$ and $\hat{F}$ are complex amplitudes.	$\hat{v}_R$ and $\hat{F}_R$ are complex amplitudes.	$\hat{U}$ and $\hat{p}$ are complex amplitudes.	$\hat{E}$ and $\hat{I}$ are complex amplitudes.
$z = r + jx$	$z_R = r_R + jx_R$	$z_A = r_A + jx_A$	$Z = R + jX$
r = responsiveness	r_R = rotational responsiveness	r_A = acoustic responsiveness	R = resistance
x = excitability	x_R = rotational excitability	x_A = acoustic excitability	X = reactance
Mechanical immobility	Rotational immobility	Acoustic immobility	Admittance
$y = 1/z = g + jb$	$y_R = 1/z_R = g_R + jb_R$	$y_A = 1/z_A = g_A + jb_A$	$Y = 1/Z = G + jB$
g = unresponsiveness	g_R = rotational unresponsiveness	g_A = acoustic unresponsiveness	G = conductance
b = unexcitability	b_R = rotational unexcitability	b_A = acoustic unexcitability	B = susceptance
$\hat{F} = \hat{v}y$	$\hat{F}_R = \hat{v}_R y_R$	$\hat{p} = \hat{U} y_A$	$\hat{I} = \hat{E}Y$
$\hat{v} = \hat{F}z$	$\hat{v}_R = \hat{F}_R z_R$	$\hat{U} = \hat{p} z_A$	$\hat{E} = \hat{I}Z$
$P = vF$	$P_i = v_R F_R$	$P_i = Up$	$P_i = EI$
$P = v\|F\| \cos \varphi_z = \|F\|^2 r$	$P = v_R\|F_R\| \cos \varphi_{zR} = \|F_R\|^2 r_R$	$P = U\|p\| \cos \varphi_{zA} = \|p\|^2 r_A$	$P = E\|I\| \cos \varphi_Z = \|I\|^2 R$

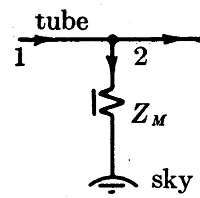
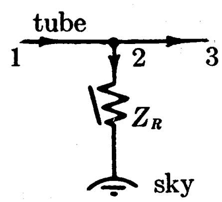
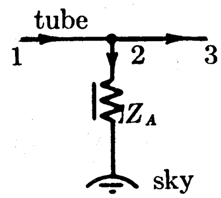
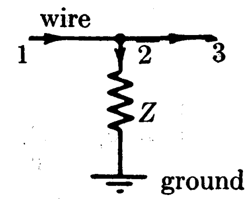
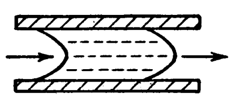
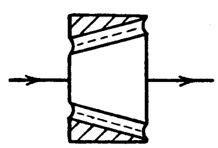
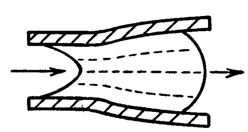
THE IMPEDANCE ANALOGY

Rectilineal mechanical	Rotational mechanical	Acoustic	Electric
		Referents	
 Sky, force of reference, (a reservoir of ideal fluid maintained at constant pressure) $F = 0$ Velocity v is measured relative to the surrounding tubing or full space (no symbol)	 Sky, torque of reference, (a reservoir of ideal fluid maintained at constant pressure) $F_R = 0$ Angular velocity v_R is measured relative to the surrounding tubing or full space (no symbol)	 Sky, pressure of reference, (usually an atmosphere of fluid at constant pressure P_0) $p = 0$ Volume velocity U is measured relative to the surrounding tubing or full space (no symbol)	 Ground, voltage of reference, (a source of charge at constant voltage) $E = 0$ Current I is measured relative to the current in the surrounding empty space (no symbol)
		Passive Elements	
 Z_M A mechanical impedance Z_M. $Z_M \equiv \dfrac{\hat{F} \text{ across}}{\hat{v} \text{ through}}$	 Z_R A rotational impedance Z_R. $Z_R \equiv \dfrac{\hat{F}_R \text{ across}}{\hat{v}_R \text{ through}}$	 Z_A An acoustic impedance Z_A. $Z_A \equiv \dfrac{\hat{p} \text{ across}}{\hat{U} \text{ through}}$	 Z An electric impedance Z. $Z \equiv \dfrac{\hat{E} \text{ across}}{\hat{I} \text{ through}}$
$Z_M = R_M + jX_M$ R_M = mechanical resistance X_M = mechanical reactance	$Z_R = R_R + jX_R$ R_R = rotational resistance X_R = rotational reactance	$Z_A = R_A \pm jX_A$ R_A = acoustic resistance X_A = acoustic reactance	$Z = R + jX$ R = resistance X = reactance
Mechanical admittance $Y_M = 1/Z_M$ $= G_M + jB_M$ G_M = mechanical conductance B_M = mechanical susceptance	Rotational admittance $Y_R = 1/Z_R$ $= G_R + jB_R$ G_R = rotational conductance B_R = rotational susceptance	Acoustic admittance $Y_A = 1/Z_A$ $= G_A + jB_A$ G_A = acoustic conductance B_A = acoustic susceptance	Admittance $Y = 1/Z = G + jB$ G = conductance B = susceptance
$\hat{v} = \hat{F} Y_M$ $\hat{F} = \hat{v} Z_M$ $P_i = Fv$ $P = F\|v\| \cos \varphi_{ZM}$ $P = v\|^2 R_M$	$\hat{v}_R = \hat{F}_R Y_R$ $\hat{F}_R = \hat{v}_R Z_R$ $P_i = F_R v_R$ $P = F_R\| v_R\| \cos \varphi_{ZR}$ $P = v_R\|^2 R_R$	$\hat{U} = \hat{p} Y_A$ $\hat{p} = \hat{U} Z_A$ $P_i = pU$ $P = p\| U\| \cos \varphi_{ZA}$ $P = U\|^2 R_A$	$\hat{I} = \hat{E} Y$ $\hat{E} = \hat{I} Z$ $P_i = EI$ $P = E\| I\| \cos \varphi_Z$ $P = I\|^2 R$

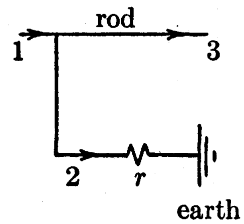
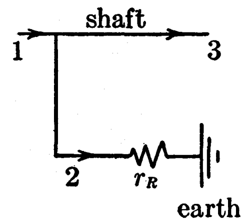
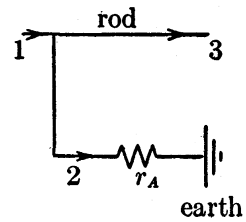
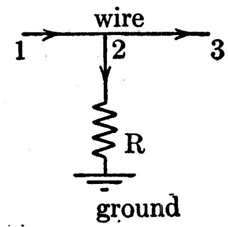
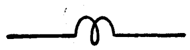
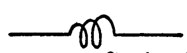
THE MOBILITY ANALOGY

Rectilineal mechanical	Rotational mechanical	Acoustic	Electric
			
Three-terminal mobility z including (rigid) junction. $\hat{v}_1 = \hat{v}_2 = \hat{v}_3 = \hat{v}$ $\hat{F}_3 = \hat{F}_1 - \hat{F}_2$ $\hat{F}_2 = \hat{v}/z$	Three-terminal rotational mobility z_R including (rigid) junction. $\hat{v}_{R1} = \hat{v}_{R2} = \hat{v}_{R3} = \hat{v}$ $\hat{F}_{R3} = \hat{F}_{R1} - \hat{F}_{R2}$ $\hat{F}_{R2} = \hat{v}_R/z_R$	Three-terminal acoustic mobility z_A including (rigid) junction. $\hat{U}_1 = \hat{U}_2 = \hat{U}_3 = \hat{U}$ $\hat{p}_3 = \hat{p}_1 - \hat{p}_2$ $\hat{p}_2 = \hat{U}/z_A$	Three-terminal impedance Z including junction. $\hat{E}_1 = \hat{E}_2 = \hat{E}_3 = \hat{E}$ $\hat{I}_3 = \hat{I}_1 - \hat{I}_2$ $\hat{I}_2 = \hat{E}/Z$
 r Mechanical responsor of responsiveness r	 r_R Rotational responsor of rotational responsiveness r_R	 r_A Acoustic responsor of acoustic responsiveness r_A	 R Resistor of resistance R
 Typical structure: viscous oil between plates attached to the terminals	 Typical structure: viscous oil between concentric rotating cones attached to the terminals	 Typical structure: viscous oil leaking through holes in piston in movable cylinder	 Typical structure: a length of resistive wire
$r \equiv \dfrac{v \text{ across}}{F \text{ through}} = \dfrac{\hat{v}}{\hat{F}}$ $z = r$ $v = Fr$ $\hat{v} = \hat{F}r$ $P = F\|^2 r = v\|^2/r$ Displacement $s = \int v\,dt = rQ_M$ where impulse $Q_M = \int F\,dt$	$r_R \equiv \dfrac{v_R \text{ across}}{F_R \text{ through}} = \dfrac{\hat{v}_R}{\hat{F}_R}$ $z_R = r_R$ $v_R = F_R r_R$ $\hat{v}_R = \hat{F}_R r_R$ $P = F_R\|^2 r_R = v_R\|^2/r_R$ Rotational displacement $\theta = \int v_R\,dt = r_R Q_R$ where torsional impulse $Q_R = \int f_R\,dt$	$r_A \equiv \dfrac{U \text{ across}}{p \text{ through}} = \dfrac{\hat{U}}{\hat{p}}$ $z_A = r_A$ $U = pr_A$ $\hat{U} = \hat{p}r_A$ $P = p\|^2 r_A = U\|^2/r_A$ Volume displacement $S = \int U\,dt = r_A Q_A$ where acoustic impulse $Q_A = \int p\,dt$	$R \equiv \dfrac{E \text{ across}}{I \text{ through}} = \dfrac{\hat{E}}{\hat{I}}$ $Z = R$ $E = IR$ $\hat{E} = \hat{I}R$ $P = I\|^2 R = E\|^2/R$ Voltage impulse $S = \int E\,dt = RQ$ where charge $Q = \int I\,dt$

THE IMPEDANCE ANALOGY

Rectilineal mechanical	Rotational mechanical	Acoustic	Electric
			
Three-terminal mechanical impedance Z_M including hydraulic junction. $\hat{F}_1 = \hat{F}_2 = \hat{F}_3 = \hat{F}$ $\hat{v}_3 = \hat{v}_1 - \hat{v}_2$ $\hat{v}_2 = \hat{F}/Z_M$	Three-terminal rotational impedance Z_R including hydraulic junction. $\hat{F}_{R1} = \hat{F}_{R2} = \hat{F}_{R3} = \hat{F}_R$ $v_{R3} = v_{R1} - v_{R2}$ $\hat{v}_{R2} = \hat{F}_R/Z_R$	Three-terminal acoustic impedance Z_A including hydraulic junction. $\hat{p}_1 = \hat{p}_2 = \hat{p}_3 = \hat{p}$ $\hat{U}_3 = \hat{U}_1 - \hat{U}_2$ $\hat{U}_2 = \hat{p}/Z_A$	Three-terminal impedance Z including junction. $\hat{E}_1 = \hat{E}_2 = \hat{E}_3 = \hat{E}$ $\hat{I}_3 = \hat{I}_1 - \hat{I}_2$ $\hat{I}_2 = \hat{E}/Z$
 R_M Mechanical resistor of resistance R_M	 R_R Rotational resistor of rotational resistance R_R	 R_A Acoustic resistor of acoustic resistance R_A	 R Resistor of resistance R
			
Typical structure: a length of viscous fluid moving in a stationary tube of unit cross-sectional area	Typical structure: an annulus of viscous liquid between a rotating cone and a stationary cone; or a coil of tubing containing viscous fluid	Typical structure: a length of viscous fluid moving in a stationary tube of any cross-sectional area	Typical structure: a length of resistive wire
$R_M \equiv \dfrac{F \text{ across}}{v \text{ through}} = \dfrac{\hat{F}}{\hat{v}}$ $Z_M = R_M$ $F = vR_M$ $\hat{F} = \hat{v}R_M$ $P = \|v\|^2 R_M = \|F\|^2/R_M$ Impulse $Q_M = \int F\,dt = R_M s$ where displacement $s = \int v\,dt$	$R_R \equiv \dfrac{F_R \text{ across}}{v_R \text{ through}} = \dfrac{\hat{F}_R}{\hat{v}_R}$ $Z_R = R_R$ $F_R = v_R R_R$ $\hat{F}_R = \hat{v}_R R_R$ $P = \|v_R\|^2 R_R = \|f_R\|^2/R_R$ Torsional impulse $Q_R = \int F_R\,dt = R_R\theta$ where displacement $\theta = \int v_R\,dt$	$R_A \equiv \dfrac{p \text{ across}}{U \text{ through}} = \dfrac{\hat{p}}{\hat{U}}$ $Z_A = R_A$ $p = UR_A$ $\hat{p} = \hat{U}R_A$ $P = \|U\|^2 R_A = \|p\|^2/R_A$ Acoustic impulse $Q_A = \int p\,dt = R_A S_A$ where volume displacement $S_A = \int U\,dt$	$R \equiv \dfrac{E \text{ across}}{I \text{ through}} = \dfrac{\hat{E}}{\hat{I}}$ $Z = R$ $E = IR$ $\hat{E} = \hat{I}R$ $P = \|I\|^2 R = \|E\|^2/R$ Voltage impulse $S = \int E\,dt = RQ$ where charge $Q = \int I\,dt$

THE MOBILITY ANALOGY

Rectilineal mechanical	Rotational mechanical	Acoustic	Electric
			
Three-terminal mechanical responsor r including junction.	Three-terminal rotational responsor r_R including junction.	Three-terminal acoustic responsor r_A including (rigid) junction.	Three-terminal resistor R including (soldered) junction.
$v_1 = v_2 = v_3 = v$ $F_3 = F_1 - F_2$ $\hat{F}_2 = \hat{v}/r$	$v_{R1} = v_{R2} = v_{R3} = v_R$ $F_{R3} = F_{R1} - F_{R2}$ $\hat{F}_{R2} = \hat{v}_R/r_R$	$U_1 = U_2 = U_3 = U$ $p_3 = p_1 - p_2$ $\hat{p}_2 = \hat{U}/r_A$	$E_1 = E_2 = E_3 = E$ $I_3 = I_1 - I_2$ $\hat{I}_2 = \hat{E}/R$
 C_M (or L_M)	 C_R (or L_R)	 C_A (or L_A)	 L
Spring of compliance C_M (or L_M)	Torsional spring of rotational compliance C_R (or L_R)	Acoustic spring of acoustic compliance C_A (or L_A)	Inductor of inductance L
$v = C_M\, dF/dt$ $z = j\omega C_M$ (or $j\omega L_M$) $\hat{v} = \hat{F}z$ $W = C_M F^2/2$ Displacement $s = \int v\, dt = C_M F$	$v_R = C_R\, dF_R/dt$ $z_R = j\omega C_R$ $\hat{v}_R = \hat{F}_R z_R$ $W = C_R F_R^2/2$ Angular displacement $\theta = \int v_R\, dt = C_R F_R$	$U = C_A\, dp/dt$ $z_A = j\omega C_A$ $\hat{U} = \hat{p} z_A$ $W = C_A p^2/2$ Volume displacement $X = \int U\, dt = C_A p$ A closed volume V of gas at pressure P_0 has $C_A = V/\gamma P_0$	$E = L\, dI/dt$ $Z = j\omega L$ $\hat{E} = \hat{I}Z$ $W = LI^2/2$ Voltage impulse $S = \int E\, dt = LI$

THE IMPEDANCE ANALOGY

<table>
<tr><th>Rectilineal mechanical</th><th>Rotational mechanical</th><th>Acoustic</th><th>Electric</th></tr>
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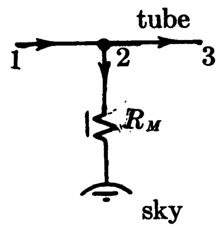
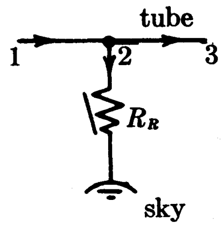
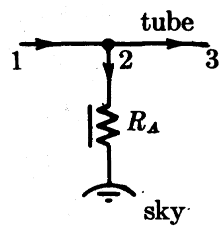
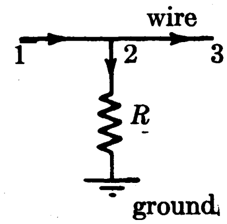
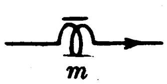
<td>

Three-terminal mechanical resistor R_M including hydraulic junction.
$F_1 = F_2 = F_3 = F$
$v_3 = v_1 - v_2$
$\hat{v}_2 = \hat{F}/R_M$</td>
<td>

Three-terminal rotational resistor R_R including hydraulic junction.
$F_{R1} = F_{R2} = F_{R3} = F_R$
$v_{R3} = v_{R1} - v_{R2}$
$\hat{v}_{R2} = \hat{F}_{R2}/R_R$</td>
<td>

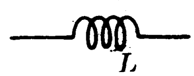
Three-terminal acoustic resistor R_A including hydraulic junction.
$p_1 = p_2 = p_3 = p$
$U_3 = U_1 - U_2$
$\hat{U}_2 = \hat{p}/R_A$</td>
<td>

Three-terminal impedance Z including (soldered) junction.
$E_1 = E_2 = E_3 = E$
$I_3 = I_1 - I_2$
$\hat{I}_2 = E/R$</td>
</tr>
<tr>
<td>

A mass, of mass m

Typical structure: a length of massive fluid contained in a unit-area tube</td>
<td>

A rotational inertor of polar moment of inertia J
Typical structure: a coil of stationary tubing containing a massive fluid</td>
<td>

An acoustic inertor of inertance M (or M_A). $M = m/A^2$
Typical structure: a mass m of gas in a stationary tube or neck of area A</td>
<td>

An inductor of inductance L

Typical structure: a coil of wire</td>
</tr>
<tr>
<td>$F = m\,dv/dt = ma$
$Z_M = j\omega m$
$\hat{F} = \hat{v}Z_M$
$W = mv^2/2$
Impulse equals momentum

$Q_M = \int F\,dt = mv$</td>
<td>$F_R = J\,dv_R/dt = J\alpha$
$Z_R = j\omega J$
$\hat{F}_R = \hat{v}_R Z_R$
$W = Jv_R^2/2$
Torsional impulse equals angular momentum
$Q_R = \int F_R\,dt = Jv_R$</td>
<td>$p = M\,dU/dt = pA_A$
$Z_A = j\omega M$
$\hat{p} = \hat{U}Z_A$
$W = MU^2/2$
Acoustic impulse equals acoustic momentum
$Q_A = \int p\,dt = MU$</td>
<td>$E = L\,dI/dt$
$Z = j\omega L$
$\hat{E} = \hat{I}Z$
$W = LI^2/2$
Voltage impulse

$S = \int E\,dt = LI$</td>
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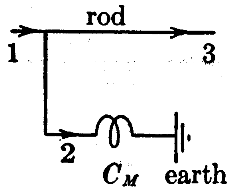
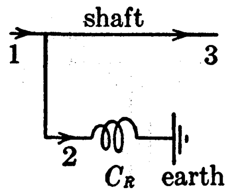
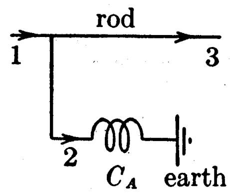
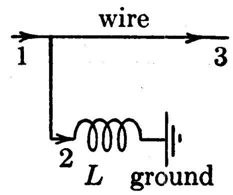
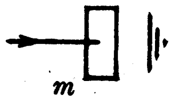
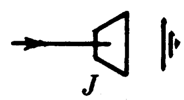
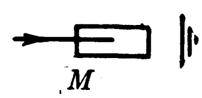
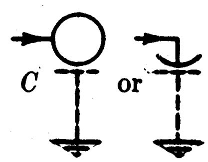
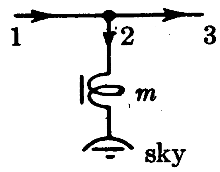
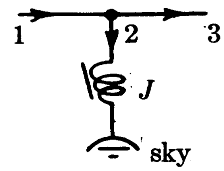
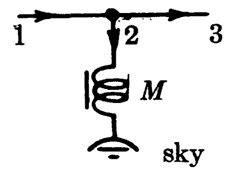
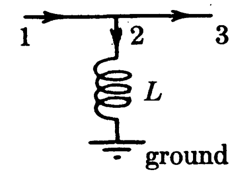
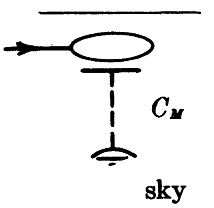
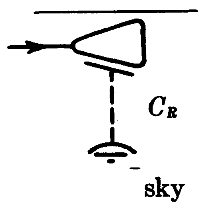
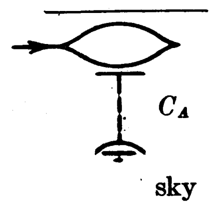
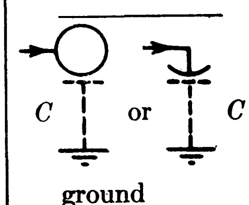
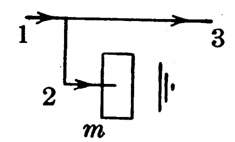
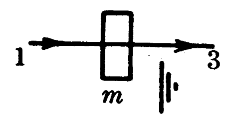
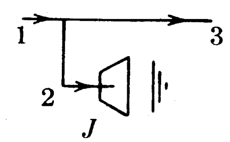
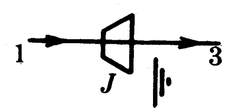
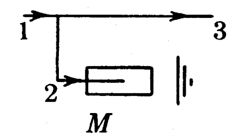
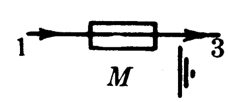
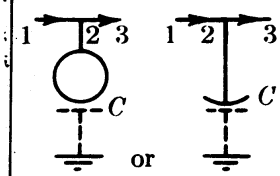
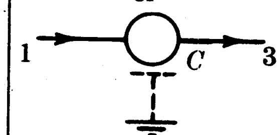
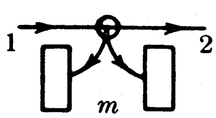
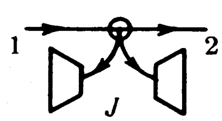
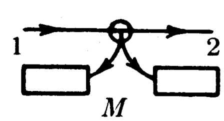
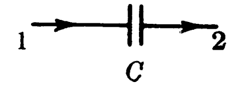
THE MOBILITY ANALOGY

Rectilineal mechanical	Rotational mechanical	Acoustic	Electric
			
Three-terminal spring (elastor) of compliance C_M, including junction.	Three-terminal torsional spring (elastor) of rotational compliance C_R, including junction.	Three-terminal acoustic spring (elastor) of acoustic compliance C_A, including junction.	Three-terminal inductor of inductance L, including junction.
$v_1 = v_2 = v_3 = v$ $F_3 = F_1 - F_2$ $\hat{F}_2 = \hat{v}/j\omega C_M$	$v_{R1} = v_{R2} = v_{R3} = v_R$ $F_{R3} = F_{R1} - F_{R2}$ $\hat{F}_{R2} = \hat{v}_R/j\omega C_R$	$U_1 = U_2 = U_3 = U$ $p_3 = p_1 - p_2$ $\hat{p}_2 = \hat{U}/j\omega C_A$	$E_1 = E_2 = E_3 = E$ $I_3 = I_1 - I_2$ $\hat{I}_2 = \hat{E}/j\omega L$
			
A mass of mass m	A rotational inertor of polar moment of inertia J	An acoustic inertor of inertance M (or M_A). $M = m/A^2$	A capacitor of capacitance C.
One terminal permanently earthed. Typical structure: a solid block constrained to lineal motion	One terminal permanently earthed. Typical structure: a flywheel	One terminal permanently earthed. Typical structure: a mass of gas in a constriction	One terminal permanently grounded. Typical structure: an isolated metal sphere in free space
$F = m\,dv/dt = ma$ $z = -j/\omega m$ $\hat{v} = \hat{F}z$ $W = mv^2/2$ Impulse equals momentum $Q_M = \int F\,dt = mv$	$F_R = J\,dv_R/dt = J\alpha$ $Z_R = -j/\omega J$ $\hat{v}_R = \hat{F}_R z_R$ $W = Jv_R^2/2$ Torsional impulse equals angular momentum $Q_R = \int F_R\,dt = Jv_R$	$p = M\,dU/dt = MA_A$ $Z_A = -j/\omega M$ $\hat{U} = \hat{p}z_A$ $W = MU^2/2$ Acoustic impulse equals acoustic momentum $Q_A = \int p\,dt = MU$	$I = C\,dE/dt$ $Z = -j/\omega C$ $\hat{E} = \hat{I}Z$ $W = CE^2/2$ Charge equals capacitance times voltage $Q = \int I\,dt = CE$

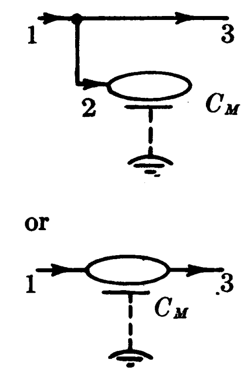
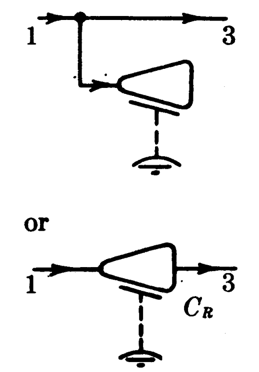
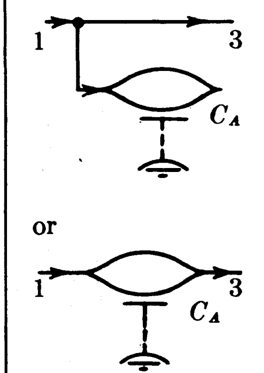
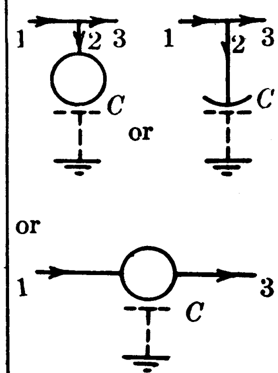
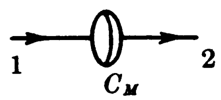
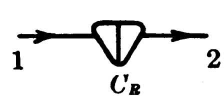
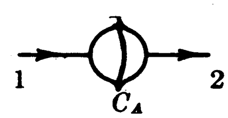
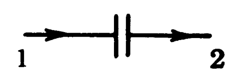
THE IMPEDANCE ANALOGY

Rectilineal mechanical	Rotational mechanical	Acoustic	Electric
			
Three-terminal mass m including hydraulic junction.	Three-terminal rotational inertor J including hydraulic junction.	Three-terminal acoustic inertor M including hydraulic junction.	Three-terminal inductor L including junction.
$F_1 = F_2 = F_3 = F$ $v_3 = v_1 - v_2$ $\hat{v}_2 = \hat{F}/Z_M$	$F_{R1} = F_{R2} = F_{R3} = F_R$ $v_{R3} = v_{R1} - v_{R2}$ $\hat{v}_{R2} = \hat{F}_R/Z_R$	$p_1 = p_2 = p_3 = p$ $U_3 = U_1 - U_2$ $\hat{U}_2 = \hat{p}/Z_A$	$E_1 = E_2 = E_3 = E$ $I_3 = I_1 - I_2$ $\hat{I}_2 = \hat{E}/Z$
			
Spring of compliance C_M. One terminal permanently skyed	Torsional spring of rotational compliance C_R. One terminal permanently skyed	Acoustic spring of acoustic compliance C_A. $C_A = V/\gamma P_0$. One terminal permanently skyed	Capacitor of capacitance C. One terminal permanently grounded
Typical structure: a bubble of gas in a rigid container, or ideal fluid in a shielded flexible container	Typical structure: a torsional spring reacting against its rigid support	Typical structure: compliant gas in a rigid container of volume V	Typical structure: an isolated metal sphere in free space
$v = C_M\, df/dt$ $Z_M = -j/\omega C_M$ $\hat{F} = \hat{v} Z_M$ $W = C_M F^2/2$ Displacement	$v_R = C_R\, dF_R/dt$ $Z_R = -j/\omega C_R$ $\hat{F}_R = \hat{v}_R Z_R$ $W = C_R F_R^2/2$ Rotational displacement	$U = C_A\, dp/dt$ $Z_A = -j/\omega C_A$ $\hat{p} = \hat{U} Z_A$ $W = C_A p^2/2$ Volume displacement	$I = C\, dE/dt$ $Z = -j/\omega C$ $\hat{E} = \hat{I} Z$ $W = CE^2/2$ Charge
$s = \int v\, dt = C_M F$	$\theta = \int v_R\, dt = C_R F_R$	$S = \int U\, dt = C_A p$	$Q = \int I\, dt = CE$

THE MOBILITY ANALOGY

Rectilineal mechanical	Rotational	Acoustic	Electric
 or 	 or 	 or 	 or 
Three-terminal mass m including junction	Three-terminal rotational inertor J including junction	Three-terminal acoustic inertor M including junction	Three-terminal capacitor C including junction
$v_1 = v_2 = v_3 = v$ $F_3 = F_1 - F_2$ $\hat{F}_2 = j\omega m\hat{v}$	$v_{R1} = v_{R2} = v_{R3} = v_R$ $F_{R3} = F_{R1} - F_{R2}$ $\hat{F}_{R2} = j\omega J\hat{v}_R$	$U_1 = U_2 = U_3 = U$ $p_3 = p_1 - p_2$ $\hat{p}_2 = j\omega M\hat{U}$	$E_1 = E_2 = E_3 = E$ $I_3 = I_1 - I_2$ $\hat{I}_2 = j\omega C\hat{E}$
			
m Transinertor of mass m	J Rotational transinertor of moment of inertia J	M Acoustic transinertor of inertance M	Condenser of capacitance C
$F_1 = F_2$ Connectible either in series with a hot rod, or to earth	$F_{R1} = F_{R2}$ Connectible either in series with a high shaft, or to earth	$p_1 = p_2$ Connectible either in series with a high rod, or to earth	$I_1 = I_2$ Connectible either in series with a high wire, or to ground
Typical structure: floating levers or a hydraulic junction which ensure equal forces in the rod and on the two masses, each of mass $2m$	Typical structure: differential gears or a hydraulic junction which ensure equal torques in the shaft and on the two inertors, each of moment of inertia $2J$	Typical structure: a hydraulic junction which ensures equal sound pressures in the main rod and on the two inertors, each of inertance $2M$	Typical structure: parallel plates between which there is a displacement current. There is no direct analogous mechanical phenomenon through *empty* space

THE IMPEDANCE ANALOGY

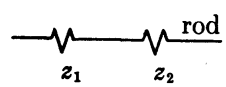
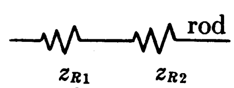
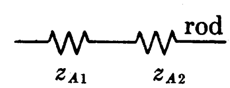
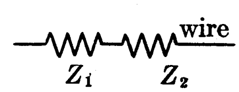
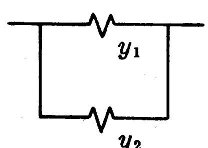
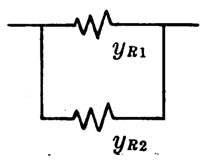
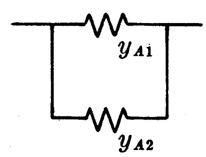
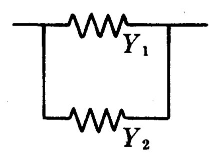
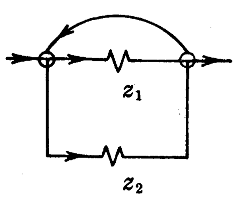
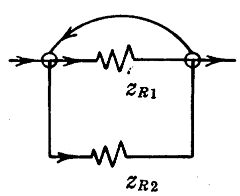
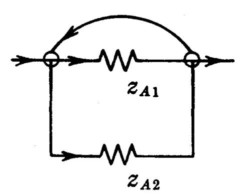
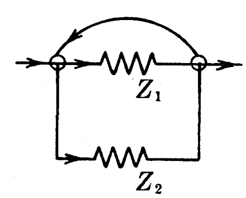
Rectilineal mechanical	Rotational mechanical	Acoustic	Electric
			
Three-terminal spring C_M including hydraulic junction. $F_1 = F_2 = F_3 = F$ $v_3 = v_1 - v_2$ $\hat{v}_2 = j\omega C_M \hat{F}$	Three-terminal torsional spring C_R including hydraulic junction. $F_{R1} = F_{R2} = F_{R3} = F_R$ $v_{R3} = v_{R1} - v_{R2}$ $\hat{v}_{R2} = j\omega C_R \hat{F}_R$	Three-terminal acoustic spring C_A including hydraulic junction. $p_1 = p_2 = p_3 = p$ $U_3 = U_1 - U_2$ $\hat{U}_2 = j\omega C_A \hat{p}$	Three-terminal capacitor including soldered junction. $E_1 = E_2 = E_3 = E$ $I_3 = I_1 - I_2$ $\hat{I}_2 = j\omega C \hat{E}$
			
Elaster of compliance C_M $v_1 = v_2$ Connectible either in series with a high tube, or to sky	Torsional elaster of rotational compliance C_R $v_{R1} = v_{R2}$ Connectible either in series with a high tube, or to sky	Acoustic elaster of acoustic compliance C_A $U_1 = U_2$ Connectible either in series with a high tube, or to sky	Condenser of capacitance C $I_1 = I_2$ Connectible either in series with a high wire, or to ground
Typical structure: a compliant diaphragm separating two chambers filled with ideal fluid	Typical structure: a torsional spring reacting against a stationary support	Typical structure: a compliant diaphragm separating two chambers filled with ideal fluid	Typical structure: parallel plates between which there is a displacement current

THE MOBILITY ANALOGY

Rectilineal mechanical	Rotational mechanical	Acoustic	Electric
If the masses are unequal $1/m = (1/m_1) + (1/m_2)$ Equations: same as for a mass (above)	If the inertors are unequal $1/J = (1/J_1) + (1/J_2)$ Equations: same as for a flywheel (above)	If the inertors are unequal $1/M = (1/M_1) + (1/M_2)$ Equations: same as for an acoustic inertor (above)	Equations: same as for a capacitor (above)

Series and Parallel Combinations of Elements

(The following combinations of general mobilities z can be extended to the pure elements. Thus, by analogy, the compliance of two springs in series is $C_M = C_{M1} + C_{M2}$ and the mass of two masses in parallel is $m = m_1 + m_2$, etc.)

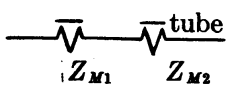
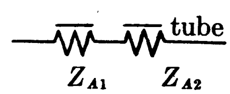
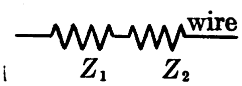
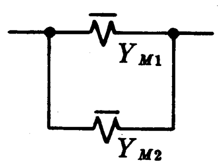
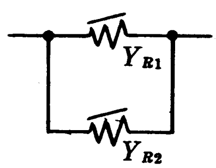
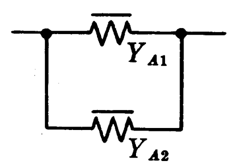
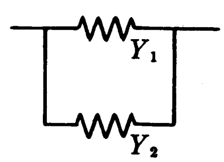
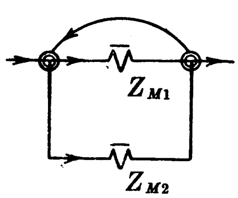
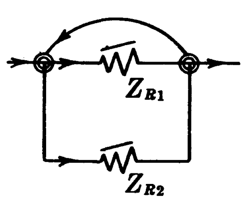
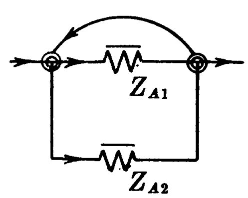
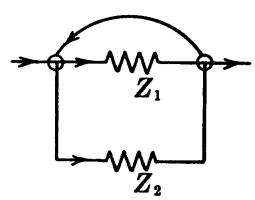
 Mobilities in series $z = z_1 + z_2$	 Rotational mobilities in series $z_R = z_{R1} + z_{R2}$	 Acoustic mobilities in series $z_A = z_{A1} + z_{A2}$	 Impedances in series $Z = Z_1 + Z_2$
 Immobilities in parallel (rigid junctions) $y = y_1 + y_2$	 Rotational immobilities in parallel (rigid junctions) $y_R = y_{R1} + y_{R2}$	 Acoustic immobilities in parallel (rigid junctions) $y_A = y_{A1} + y_{A2}$	 Admittances in parallel (soldered junctions) $Y = Y_1 + Y_2$
 Mobilities enmeshed by hydraulic meshers $z = z_1 + z_2$ $F_1 = F_2 = F$	 Rotational mobilities enmeshed by hydraulic meshers $z_R = z_{R1} + z_{R2}$ $F_{R1} = F_{R2} = F_R$	 Acoustic mobilities enmeshed by hydraulic meshers $z_A = z_{A1} + z_{A2}$ $p_1 = p_2 = p$	 Impedances enmeshed by meshers $Z = Z_1 + Z_2$ $I_1 = I_2 = I$

THE IMPEDANCE ANALOGY

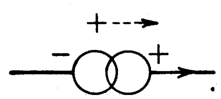
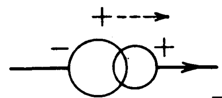
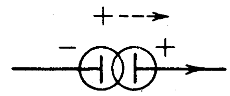
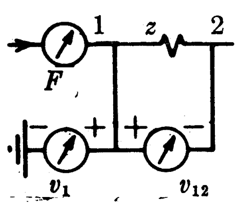
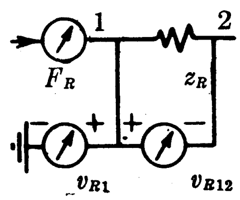
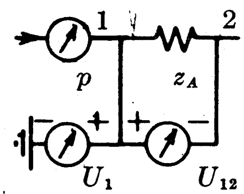
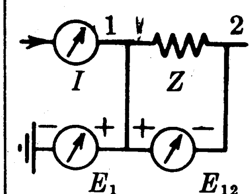
Rectilineal mechanical	Rotational mechanical	Acoustic	Electric
Equations: same as for a spring (above)	Equations: same as for a torsional spring (above)	Equations: same as for an acoustic spring (above)	Equations: same as for a capacitor (above)

Series and Parallel Combinations of Elements

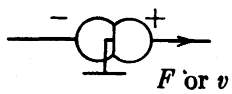
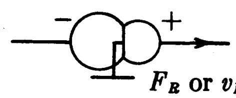
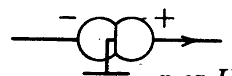
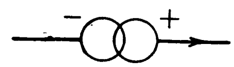
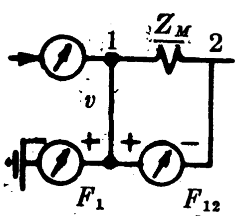
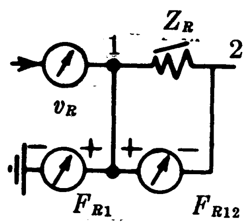
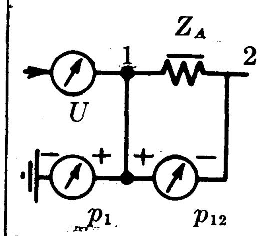
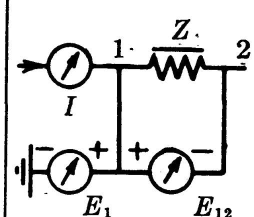
(The following combinations of general impedances can be extended to the pure elements. Thus, by analogy, the mass of two masses in series is $m = m_1 + m_2$ and the compliance of two elasters in parallel is $C_M = C_{M1} + C_{M2}$, etc.)

 Mechanical impedances in series $Z_M = Z_{M1} + Z_{M2}$	 Rotational impedances in series $Z_R = Z_{R1} + Z_{R2}$	 Acoustic impedances in series $Z_A = Z_{A1} + Z_{A2}$	 Impedances in series $Z = Z_1 + Z_2$
 Mechanical admittances in parallel (hydraulic junctions) $Y_M = Y_{M1} + Y_{M2}$	 Rotational admittances in parallel (hydraulic junctions) $Y_R = Y_{R1} + Y_{R2}$	 Acoustic admittances in parallel (hydraulic junctions) $Y_A = Y_{A1} + Y_{A2}$	 Admittances in parallel (soldered junctions) $Y = Y_1 + Y_2$
 Mechanical impedances enmeshed by rigid meshers $Z_M = Z_{M1} + Z_{M2}$ $v_1 = v_2 = v$	 Rotational impedances enmeshed by rigid meshers $Z_R = Z_{R1} + Z_{R2}$ $v_{R1} = v_{R2} = v_R$	 Acoustic impedances enmeshed by rigid meshers $Z_A = Z_{A1} + Z_{A2}$ $U_1 = U_2 = U$	 Impedances enmeshed by meshers $Z = Z_1 + Z_2$ $I_1 = I_2 = I$

THE MOBILITY ANALOGY

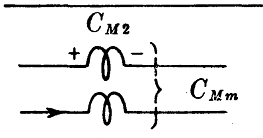
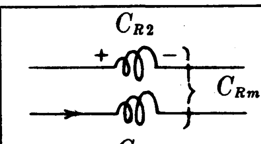
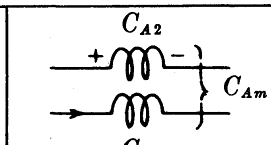
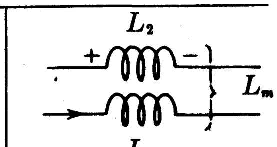
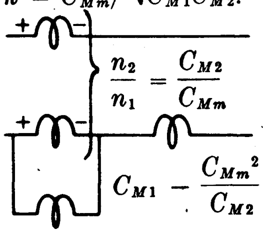
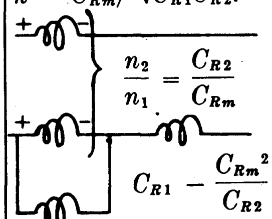
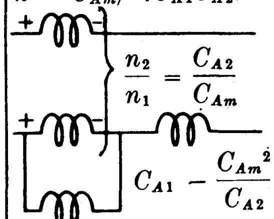
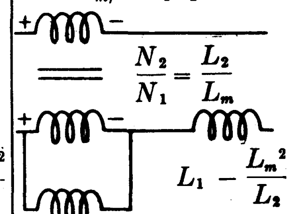
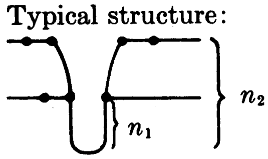
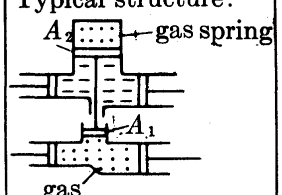
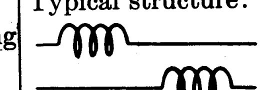
Rectilineal mechanical	Rotational mechanical	Acoustic	Electric
Vibrators			Generators
 v or F A vibrator having a vibromotive velocity v or vibromotive force F as marked	 v_R or F_R A rotational vibrator having a rotatomotive velocity v_R or rotatomotive torque F_R as marked	 U or p An acoustic vibrator having an acoustomotive volume velocity U or acoustomotive sound pressure p as marked	 E or I A generator having an electromotive force E or current I as marked
Meters and Their Connections			
 Meters with rigid junctions measure: Force F through or into z Velocity v_1 at or of terminal 1 Velocity v_{12} across z $z = \hat{v}_{12}/\hat{F}$	 Meters with rigid junctions measure: Torque F_R through or into z_R Velocity v_{R1} at or of terminal 1 Angular velocity v_{R12} across z_R $z_R = \hat{v}_{R12}/\hat{F}_R$	 Meters with rigid junctions measure: Sound pressure p through or into z_A Volume velocity U_1 at or of terminal 1 Volume velocity U_{12} across z_A $z_A = \hat{U}_{12}/\hat{p}$	 Meters with soldered junctions measure: Current I through or into Z Voltage E_1 at or of terminal 1 Voltage E_{12} across Z $Z = \hat{E}_{12}/\hat{I}$

THE IMPEDANCE ANALOGY

Rectilineal mechanical	Rotational mechanical	Acoustic	Electric
Vibrators			Generators
 F or v A vibrator pumping a vibromotive force F or velocity v as marked	 F_R or v_R A rotational vibrator having a rotatomotive torque F_R or angular velocity v_R as marked	 p or U An acoustic vibrator having an acoustomotive sound pressure p or volume velocity U as marked	 E or I A generator having an electromotive force E or current I as marked
Meters and Their Connections			
 Meters with hydraulic junctions measure: Velocity v through or into Z_M Force F_1 at or of terminal 1 Force F_{12} across Z_M $Z_M = \hat{F}_{12}/\hat{v}$	 Meters with hydraulic junctions measure: Angular velocity v_R through or into Z_R Torque F_{R1} at or of terminal 1 Torque F_{R12} across Z_R $Z_R = \hat{F}_{R12}/\hat{v}_R$	 Meters with hydraulic junctions measure Volume velocity U through or into Z_A Sound pressure p_1 at or of terminal 1 Sound pressure p_{12} across Z_A $Z_A = \hat{p}_{12}/\hat{U}$	 Meters with soldered junctions measure: Current I through or into Z Voltage E_1 at or of terminal 1 Voltage E_{12} across Z $Z = \hat{E}_{12}/\hat{I}$

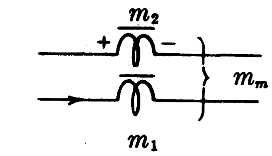
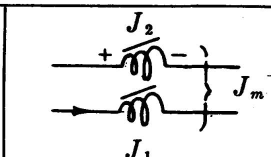
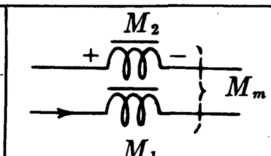
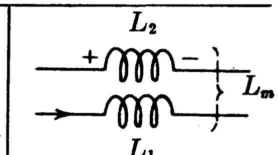
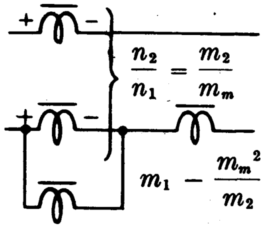
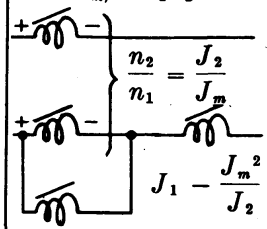
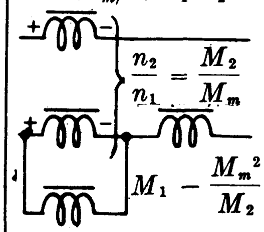
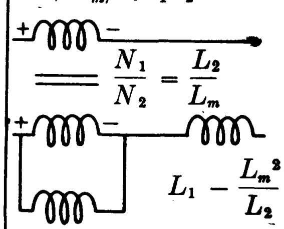
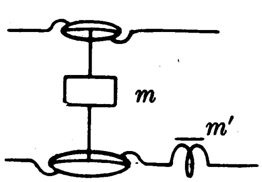
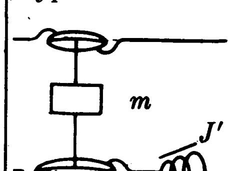
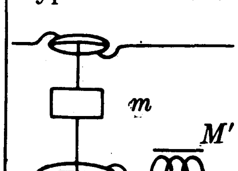
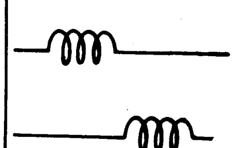
Erratum: The lower left symbol in each of the above diagrams in the first three columns should be a sky, not an earth.

THE MOBILITY ANALOGY

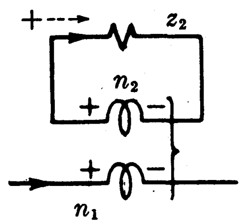
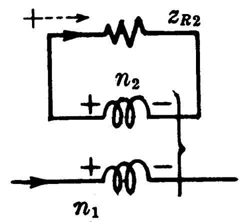
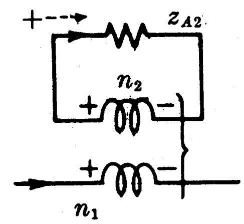
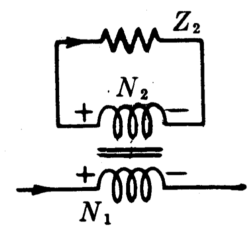
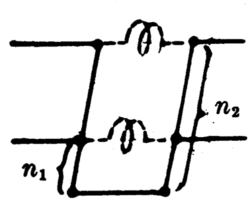
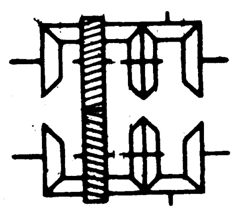
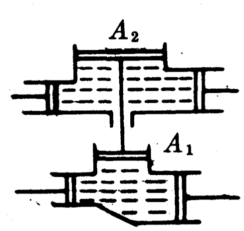
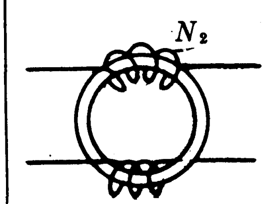
Rectilineal mechanical	Rotational mechanical	Acoustic	Electric
Mutual Couplers. Only analogues of mutual inductance are shown though analogues of mutual resistance and mutual capacitance can be built using ideal transformers			
			
Mutual compliance C_{Mm} between springs of self-compliances C_{M1} and C_{M2}.	Mutual rotational compliance C_{Rm} between torsional springs of self-compliances C_{R1} and C_{R2}.	Mutual acoustic compliance C_{Am} between acoustic springs of self-compliances C_{A1} and C_{A2}.	Mutual inductance L_m between coils of self-inductances L_1 and L_2.
$v_2 = C_{Mm}\,dF_1/dt$ $v_1 = C_{Mm}\,dF_2/dt$ $\hat{v}_2 = -j\omega C_{Mm}\hat{F}_1$ etc.	$v_{R2} = C_{Rm}\,dF_{R1}/dt$ $v_{R1} = C_{Rm}\,dF_{R2}/dt$ $\hat{v}_{R2} = -j\omega C_{Rm}\hat{F}_{R1}$ etc.	$U_2 = C_{Am}\,dp_1/dt$ $U_1 = C_{Am}\,dp_2/dt$ $\hat{U}_2 = -j\omega C_{Am}\hat{p}_1$	$E_2 = L_m\,dI_1/dt$ $E_1 = L_m\,dI_2/dt$ $\hat{E}_2 = -j\omega L_m\hat{I}_1$ etc.
Coefficient of coupling $k = C_{Mm}/\sqrt{C_{M1}C_{M2}}$.	Coefficient of coupling $k = C_{Rm}/\sqrt{C_{R1}C_{R2}}$.	Coefficient of coupling $k = C_{Am}/\sqrt{C_{A1}C_{A2}}$.	Coefficient of coupling $k = L_m/\sqrt{L_1L_2}$.
			
C_{Mm}^2/C_{M2} Ideal transformer with springs, equivalent to mutual springs above.	C_{Rm}^2/C_{R2} Ideal transformer with springs, equivalent to mutual springs above.	C_{Am}^2/C_{A2} Ideal transformer with acoustic springs, equivalent to mutual springs above.	L_m^2/L_2 Ideal transformer with inductors, equivalent to mutual inductors above.
Typical structure:  A compliant hairpin with lever ratio n_2/n_1.	Typical structure: As above; springs in series and parallel with primary differential, geared to secondary differential.	Typical structure:  Gas spring on coupling member between fluid filled differentials.	Typical structure:  Proximate coils with air or iron core.

THE IMPEDANCE ANALOGY

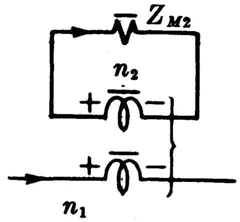
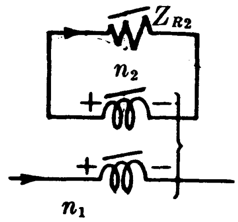
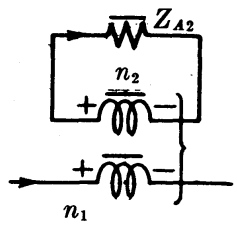
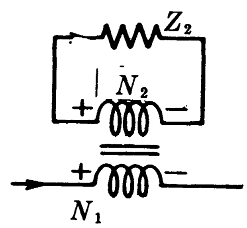
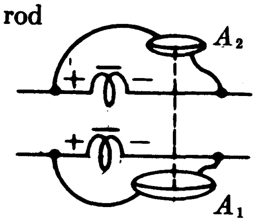
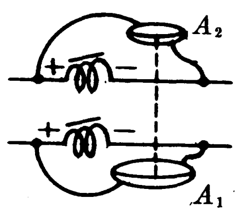
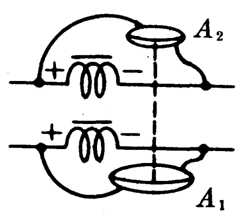
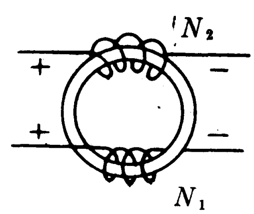
Mutual Couplers. Only analogues of mutual inductance are shown though analogues of mutual resistance and mutual capacitance can be built using ideal transformers

Rectilineal mechanical	Rotational mechanical	Acoustic	Electric
 	 	 	
Mutual mass m_m between masses of self-masses m_1 and m_2.	Mutual moment of inertia J_m between rotational inertors of moments of inertia J_1 and J_2.	Mutual inertance M_m between acoustic inertors of inertances M_1 and M_2.	Mutual inductance L_m between coils of self-inductances L_1 and L_2.
$F_2 = m_m\, dv_1/dt$ $F_1 = m_m\, dv_2/dt$ $\hat{F}_2 = -j\omega m_m \hat{v}_1$ etc. Coefficient of coupling $k = m_m/\sqrt{m_1 m_2}$.	$F_{R2} = J_m\, dv_{R1}/dt$ $F_{R1} = J_m\, dv_{R2}/dt$ $\hat{F}_{R2} = -j\omega J_m \hat{v}_{R1}$ etc. Coefficient of coupling $k = J_m/\sqrt{J_1 J_2}$.	$p_2 = M_m\, dU_1/dt$ $p_1 = M_m\, dU_2/dt$ $\hat{p}_2 = -j\omega M_m \hat{U}_1$ etc. Coefficient of coupling $k = M_m/\sqrt{M_1 M_2}$.	$E_2 = L_m\, dI_1/dt$ $E_1 = L_m\, dI_2/dt$ $\hat{E}_2 = -j\omega L_m \hat{I}_1$ etc. Coefficient of coupling $k = L_m/\sqrt{L_1 L_2}$.
 	 	 	
m_m^2/m_2 Ideal transformer with masses, equivalent to mutual masses above.	J_m^2/J_2 Ideal transformer with inertors, equivalent to mutual inertors above.	M_m^2/M_2 Ideal transformer with acoustic inertors, equivalent to mutual inertors above.	L_m^2/L_2 Ideal transformer with inductors, equivalent to mutual inductors above.
Typical structure:  	Typical structure:  	Typical structure:  	Typical structure: 
Inertia on connector between differentials, plus series mass m'.	Inertia on connector between differentials, plus series inertor J'.	Inertia on connector between differentials, plus series inertor M'.	Proximate coils with air or iron core.

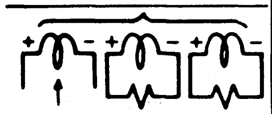
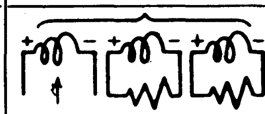
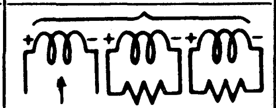
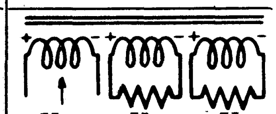
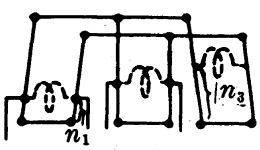
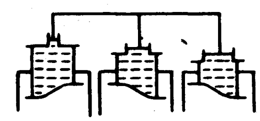
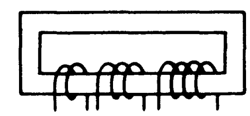
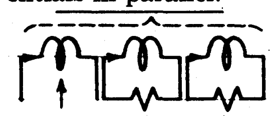
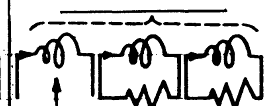
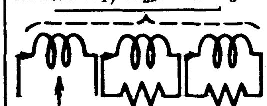
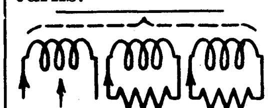
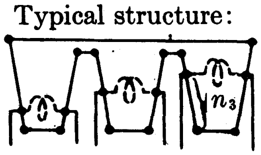
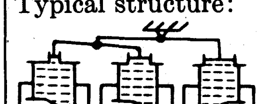
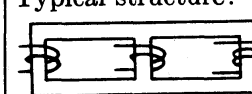
THE MOBILITY ANALOGY

Rectilineal	Rotational	Acoustic	Electric
Ideal Transformers			
			
Lineal transformer with load The bracketed springs of great compliance symbolize the coupling of the velocities *across* primary and secondary, and the force *through* the rods, by means of levers shown below.	Rotational transformer with load The bracketed springs of great compliance symbolize the coupling of the angular velocities *across* primary and secondary and the torque *through* the shafts, by means of gears shown below.	Acoustic transformer with load The bracketed acoustic springs of great compliance symbolize the coupling of the volume velocities *across* primary and secondary, and the sound pressures *through* the rods, by means of pistons below.	Electric transformer with load The inductors of great inductance symbolize the coupling of the voltages *across* the primary and the secondary, and the currents *through* the wires, by means of the iron core below.
$v_2 = n_2v_1/n_1$ $F_1 = n_2F_2/n_1$ $z_1 = z_2(n_1/n_2)^2$ Typical structure: Hinged floating levers multiply the velocities across primary and secondary	$v_{R2} = n_2v_{R1}/n_1$ $F_{R1} = n_2F_{R2}/n_1$ $z_{R1} = z_{R2}(n_1/n_2)^2$ Typical structure: Primary and secondary differentials with propeller shafts geared together	$U_2 = n_2U_1/n_1$ $p_1 = n_2p_2/n_1$ $z_{A1} = z_{A2}(n_1/n_2)^2$ Typical structure: Hydraulic differentials with cross-connected pistons	$E_2 = N_2E_1/N_1$ $I_1 = N_2I_2/N_1$ $Z_1 = Z_2(N_1/N_2)^2$ Typical structure: Two coils wound on a laminated iron ring
			
n_2/n_1 is lever ratio	n_2/n_1 is gear ratio	$n_2/n_1 = A_2/A_1$	N_2/N_1 is turns ratio
Very compliant dotted spring *s may* pass *constant* primary force with no secondary force; not ideal at zero frequency·	Analogous possibility	Analogous possibility	The primary *may* pass a *constant* current with no secondary current; not ideal at zero frequency

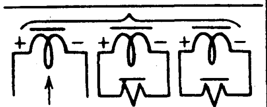
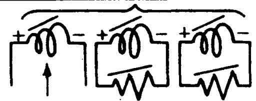
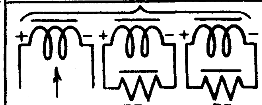
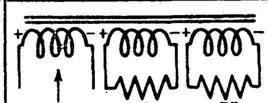
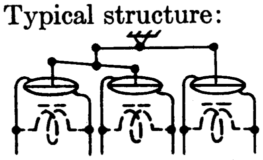
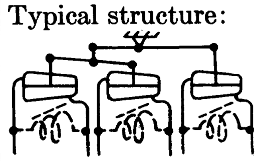
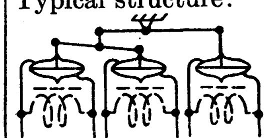
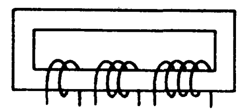
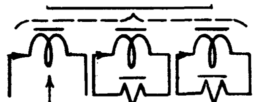
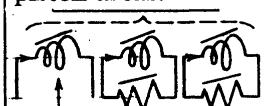
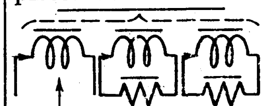
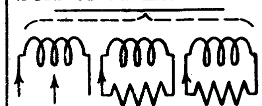
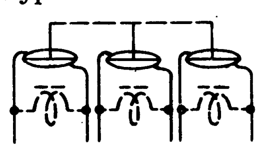
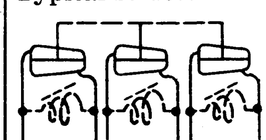
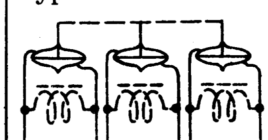
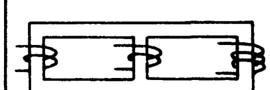
THE IMPEDANCE ANALOGY

Rectilineal	Rotational	Acoustic	Electric
Ideal Transformers			
			
Lineal transformer with load The bracketed masses of great mass symbolize the coupling of the forces *across* the primary and secondary, and the velocities *through* the tubes, by auxiliary mechanism as shown below.	Rotational transformer with load The bracketed rotational inertors of great moments of inertia symbolize the coupling of the torques *across* the primary and secondary, and the angular velocities *through* the tubes, by auxiliary mechanism as shown below.	Acoustic transformer with load The bracketed acoustic inertors of great inertances symbolize the coupling of the sound pressures *across* the primary and secondary, and the volume velocities *through* the tubes, by auxiliary mechanism as shown below.	Electric transformer with load The inductors of great inductance symbolize the coupling of the voltages *across* the primary and secondary, and the currents *through* the wires, by an iron core shown below.
$F_2 = n_2F_1/n_1$ $v_1 = n_2v_2/n_1$ $Z_{M1} = Z_{M2}(n_1/n_2)^2$ Typical structure: Two differential diaphragms or pistons, of different areas, actuated by the forces across primary and secondary, are connected by the dotted rod	$F_{R2} = n_2F_{R1}/n_1$ $v_{R1} = n_2v_{R2}/n_1$ $Z_{R1} = Z_{R2}(n_1/n_2)^2$ Typical structure: Two differentials actuated by the torques across primary and secondary are rigidly interconnected by the dotted shaft	$p_2 = n_2p_1/n_2$ $U_1 = n_2U_2/n_1$ $Z_{A1} = Z_{A2}(n_1/n_2)^2$ Typical structure: Same as the rectilineal transformer but the terminal tubes are of any areas	$E_2 = N_2E_1/N_1$ $I_1 = N_2I_2/N_1$ $Z_1 = Z_2(N_1/N_2)^2$ Typical structure: Two coils wound on an iron ring
			
$(n_1/n_2) = (A_2/A_1)$ The large mass(es) *may* pass *constant* primary velocity with no secondary velocity; not necessarily ideal at zero frequency	$(n_1/n_2) = (A_2/A_1)$ Analogous possibility	$(n_1/n_2) = (A_2/A_1)$ Analogous possibility	N_1/N_2 is turns ratio The primary *may* pass a *constant* current with no secondary current, not ideal at zero frequency

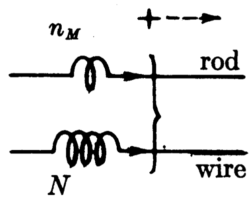
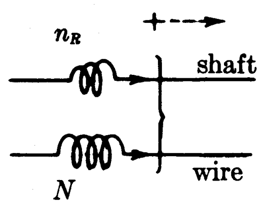
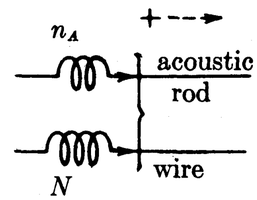
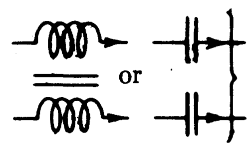
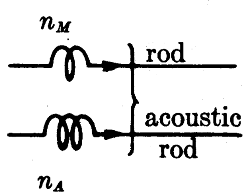
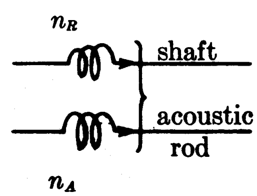
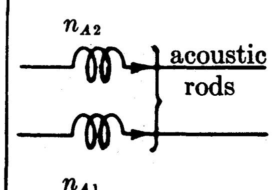
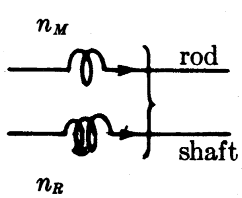
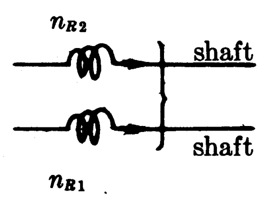
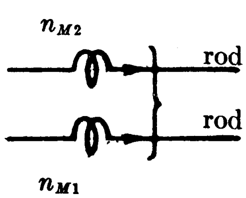
THE MOBILITY ANALOGY

Rectilineal mechanical	Rotational mechanical	Acoustic	Electric
 y_1 y_2 y_3	 y_{R1} y_{R2} y_{R3}	 y_{A1} y_{A2} y_{A3}	 Y_1 Y_2 Y_3
Multiple element velocity transformer with loads.	Multiple element angular velocity transformer with loads.	Multiple element volume velocity transformer with loads.	Multiple element voltage transformer with loads.
$\frac{v_1}{n_1} = \frac{v_2}{n_2} = \frac{v_3}{n_3}$ $y_1 = \left(\frac{n_2}{n_1}\right)^2 y_2 + \left(\frac{n_3}{n_1}\right)^2 y_3$	$\frac{v_{R1}}{n_1} = \frac{v_{R2}}{n_2} = \frac{v_{R3}}{n_3}$ $y_{R1} = \left(\frac{n_2}{n_1}\right)^2 y_{R2} + \left(\frac{n_3}{n_1}\right)^2 y_{R3}$	$\frac{U_1}{n_1} = \frac{U_2}{n_2} = \frac{U_3}{n_3}$ $y_{A1} = \left(\frac{n_2}{n_1}\right)^2 y_{A2} + \left(\frac{n_3}{n_1}\right)^2 y_{A3}$	$\frac{E_1}{N_1} = \frac{E_2}{N_2} = \frac{E_3}{N_3}$ $Y_1 = \left(\frac{N_2}{N_1}\right)^2 Y_2 + \left(\frac{N_3}{N_1}\right)^2 Y_3$
Typical structure:  Hinged lever differentials in parallel.	Typical structure. A multiplicity of differentials with their propeller shafts geared together	Typical structure:  Hydraulic differentials with top piston areas n_1, n_2 and n_3.	Typical structure:  Ring core with coils of N_1, N_2 and N_3 turns.
 z_1 z_2 z_3	 z_{R1} z_{R2} z_{R3}	 z_{A1} z_{A2} z_{A3}	 Z_1 Z_2 Z_3
Multiple element force transformer with loads.	Multiple element torque transformer with loads.	Multiple element sound pressure transformer with loads.	Multiple element current transformer with loads.
$n_1F_1 = n_2F_2 = n_3F_3$ $z_1 = \left(\frac{n_1}{n_2}\right)^2 z_2 + \left(\frac{n_1}{n_3}\right)^2 z_3$	$n_1F_{R1} = n_2F_{R2} = n_3F_{R3}$ $z_{R1} = \left(\frac{n_1}{n_2}\right)^2 z_{R2} + \left(\frac{n_1}{n_3}\right)^2 z_{R3}$	$n_1p_1 = n_2p_2 = n_3p_3$ $z_{A1} = \left(\frac{n_1}{n_2}\right)^2 z_{A2} + \left(\frac{n_1}{n_3}\right)^2 z_{A3}$	$N_1I_1 = N_2I_2 = N_3I_3$ $Z_1 = \left(\frac{N_1}{N_2}\right)^2 Z_2 + \left(\frac{N_1}{N_3}\right)^2 Z_3$
Typical structure:  Hinged lever differentials in series.	Typical structure: A multiplicity of differentials with propeller shafts geared through differentials.	Typical structure:  Hydraulic differentials with whiffletrees. Top piston areas are n_1, n_2, and n_3.	Typical structure:  Parallel leg core with coils of N_1, N_2 and N_3 turns.

THE IMPEDANCE ANALOGY

Rectilineal mechanical	Rotational mechanical	Acoustic	Electric
 Y_{M1} Y_{M2} Y_{M3} Multiple element force transformer with loads.	 Y_{R1} Y_{R2} Y_{R3} Multiple element torque transformer with loads.	 Y_{A1} Y_{A2} Y_{A3} Multiple element sound pressure transformer with loads.	 Y_1 Y_2 Y_3 Multiple element voltage transformer with loads.
$\frac{F_1}{n_1} = \frac{F_2}{n_2} = \frac{F_3}{n_3}$ $Y_{M1} = \left(\frac{n_2}{n_1}\right)^2 Y_{M2} + \left(\frac{n_3}{n_1}\right)^2 Y_{M3}$	$\frac{F_{R1}}{n_1} = \frac{F_{R2}}{n_2} = \frac{F_{R3}}{n_3}$ $Y_{R1} = \left(\frac{n_2}{n_1}\right)^2 Y_{R2} + \left(\frac{n_3}{n_1}\right)^2 Y_{R3}$	$\frac{p_1}{n_1} = \frac{p_2}{n_2} = \frac{p_3}{n_3}$ $Y_{A1} = \left(\frac{n_2}{n_1}\right)^2 Y_{A2} + \left(\frac{n_3}{n_1}\right)^2 Y_{A3}$	$\frac{E_1}{N_1} = \frac{E_2}{N_2} = \frac{E_3}{N_3}$ $Y_1 = \left(\frac{N_2}{N_1}\right)^2 Y_2 + \left(\frac{N_3}{N_1}\right)^2 Y_3$
Typical structure:  The n's are inversely proportional to piston areas.	Typical structure:  The n's are inversely proportional to the piston areas.	Typical structure:  The n's are inversely proportional to the piston areas.	Typical structure:  The N's are proportional to the numbers of turns.
 Z_{M1} Z_{M2} Z_{M3} Multiple element velocity transformer with loads.	 Z_{R1} Z_{R2} Z_{R3} Multiple element angular velocity transformer with loads.	 Z_{A1} Z_{A2} Z_{A3} Multiple element volume velocity transformer with loads.	 Z_1 Z_2 Z_3 Multiple element current transformer with loads.
$n_1v_1 = n_2v_2 = n_3v_3$ $Z_{M1} = \left(\frac{n_1}{n_2}\right)^2 Z_{M2} + \left(\frac{n_1}{n_3}\right)^2 Z_{M3}$	$n_1v_{R1} = n_2v_{R2} = n_3v_{R3}$ $Z_{R1} = \left(\frac{n_1}{n_2}\right)^2 Z_{R2} + \left(\frac{n_1}{n_3}\right)^2 Z_{R3}$	$n_1U_1 = n_2U_2 = n_3U_3$ $Z_{A1} = \left(\frac{n_1}{n_2}\right)^2 Z_{A2} + \left(\frac{n_1}{n_3}\right)^2 Z_{A3}$	$N_1I_1 = N_2I_2 = N_3I_3$ $Z_1 = \left(\frac{N_1}{N_2}\right)^2 Z_2 + \left(\frac{N_1}{N_3}\right)^2 Z_3$
Typical structure:  The n's are inversely proportional to piston areas.	Typical structure:  The n's are inversely proportional to the piston areas.	Typical structure:  The n's are inversely proportional to the piston areas.	Typical structure:  The N's are proportional to the numbers of turns.

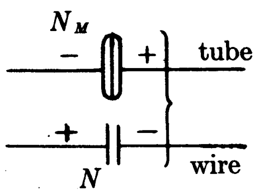
THE MOBILITY ANALOGY

Rectilineal mechanical	Rotational mechanical	Acoustic	Electric
The Conservative Ideal Direct Transducers and Transformers, Which Couple Through Variables to Through Variables, and Across Variables to Across Variables			
 Electromechanical current-force transducer $E/N = v/n_M$ $NI = n_M F$ $Z = z(N/n_M)^2$ N/n_M may be in newtons per amp or volts per m/sec	 Electrorotational current-torque transducer $E/N = v_R/n_R$ $NI = n_R F_R$ $Z = z_R(N/n_R)^2$ N/n_R may be in newton m per amp or volts per radian/sec	 Electroacoustic current-sound pressure transducer $E/N = U/n_A$ $NI = n_A p$ $Z = z_A(N/n_A)^2$ N/n_A may be in newtons/m² per amp or volts per m³/sec	 Electric transformer $E_1/N_1 = E_2/N_2$ $N_1 I_1 = N_2 I_2$ $Z_1 = Z_2(N_1/N_2)$ N_1/N_2 is the transformer ratio
 Acoustomechanical sound pressure-force transducer	 Acoustorotational sound pressure-torque transducer	 Acoustic transformer	
 Rotato-rectilineal torque-force transducer	 Rotational transformer	All these transducers and transformers have equations and units of the form given in the top row	
 Mechanical transformer			

THE IMPEDANCE ANALOGY

Rectilineal mechanical	Rotational mechanical	Acoustic	Electric

The Conservative Ideal Direct Transducers and Transformers, Which Couple Through Variables to Through Variables, and Across Variables to Across Variables



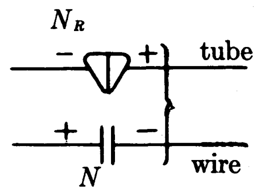
Electromechanical voltage-force transducer

$E/N = F/N_M$

$NI = N_M v$

$Z = Z_M(N/N_M)^2$

N/N_M may be in volts per newton or m/sec per amp



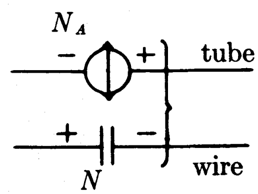
Electrorotational voltage-torque transducer

$E/N = F_R/N_R$

$NI = N_R v_R$

$Z = Z_R(N/N_R)^2$

N/N_M may be in volts per newton m or radians/sec per amp



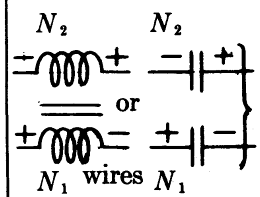
Electroacoustic voltage-sound pressure transducer

$E/N = p/N_A$

$NI = N_A U$

$Z = Z_A(N/N_A)^2$

N/N_A may be in volts per newton/m² or m²/sec per amp



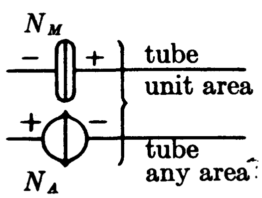
Electric (voltage-voltage) transformer

$E_1/N_1 = E_2/N_2$

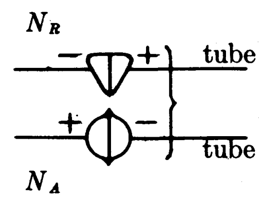
$N_1 I_1 = N_2 I_2$

$Z_1 = Z_2(N_1/N_2)^2$

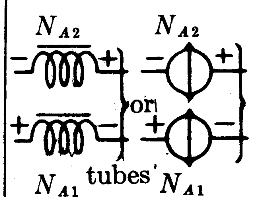
N_1/N_2 is the transformer ratio



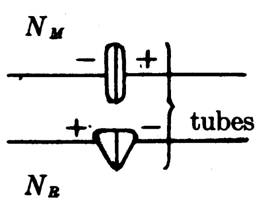
Acoustomechanical sound pressure-force transducer



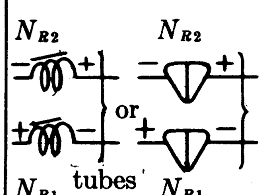
Acoustorotational sound pressure-torque transducer



Acoustic transformer



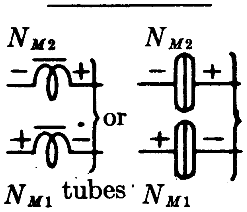
Rotato-rectilineal torque-force transducer



Rotational transducer

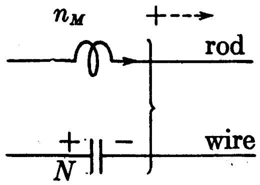
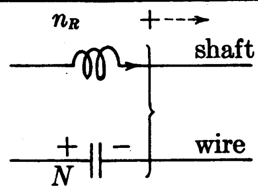
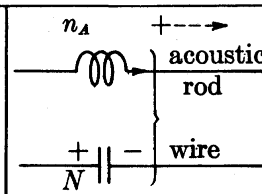
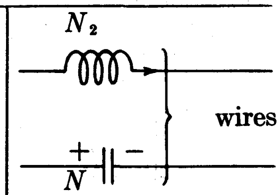
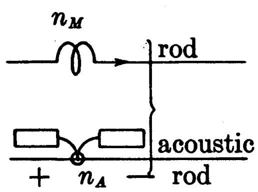
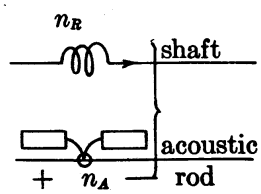
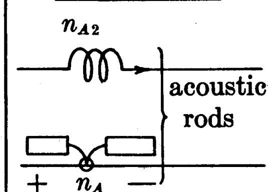
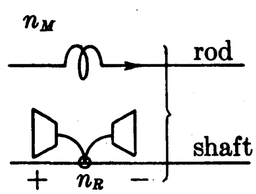
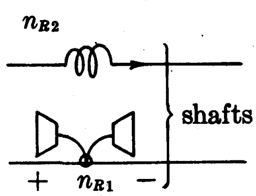
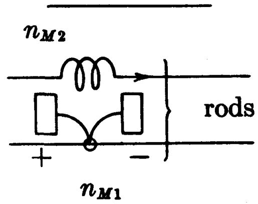
All these transducers and transformers have equations and units of the form given in the top row.

Two styles of symbol are rational for each of these elements since mutual-capacitance transformers and mutual-inductance transformers are equivalent in function

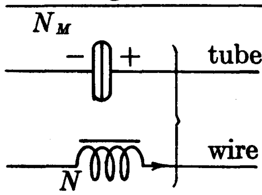
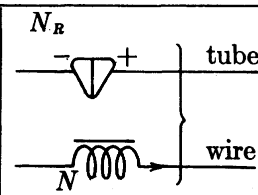
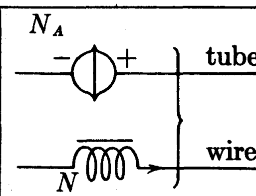
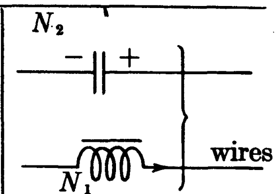
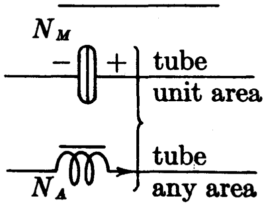
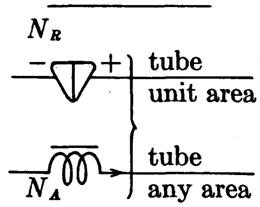
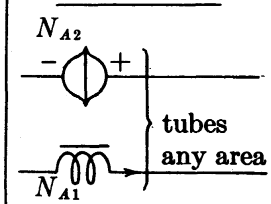
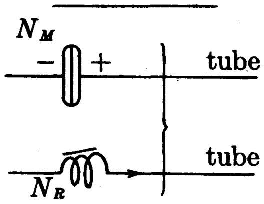
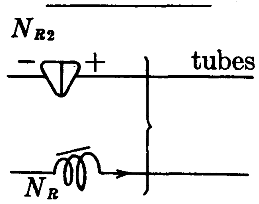
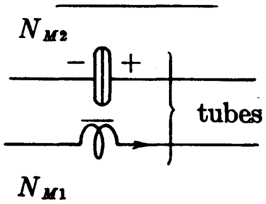


Mechanical transformer

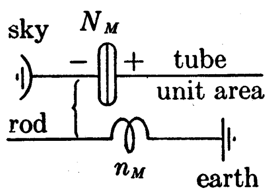
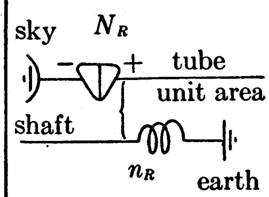
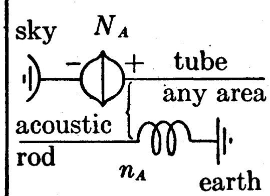
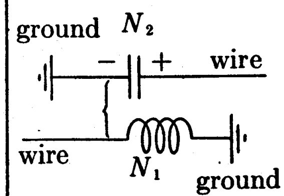
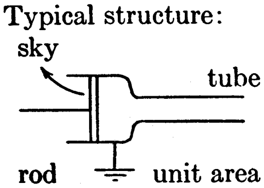
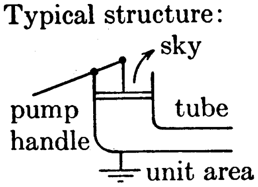
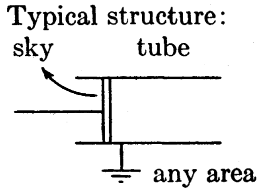
THE MOBILITY ANALOGY

Rectilineal mechanical	Rotational mechanical	Acoustic	Electric
The Conservative Ideal Inverse Transducers and Transformers, Which Couple Through Variables to Across Variables			
 Electromechanical voltage-force inverse transducer $E/N = n_M F$ $NI = v/n_M$ $Z = y(Nn_M)^2$ Nn_M may be in volts per newton or m/sec per amp	 Electrorotational voltage-torque inverse transducer $E/N = n_R F_R$ $NI = v_R/n_R$ $Z = y_R(Nn_R)^2$ Nn_R may be in volts per newton m or radian/sec per amp	 Electroacoustic voltage-sound pressure inverse transducer $E/N = n_A p$ $NI = U/n_A$ $Z = y_A(Nn_A)^2$ Nn_A may be in volts per newton/m² or m³/sec per amp	 Electric voltage-current inverse transformer. A gyrator $E_1/N_1 = N_2 I_2$ $N_1 I_1 = E_2/N_2$ $Z_1 = Y_2(N_1 N_2)^2$ $N_1 N_2$ is the transfer impedance in ohms, or volts/amp
 Acoustomechanical volume velocity-force inverse transducer	 Acoustorotational volume velocity-torque inverse transducer	 Acoustic volume velocity-sound pressure inverse transformer. An acoustic gyrator	
 Rotato-rectilineal angular velocity-force inverse transducer	 Rotational angular velocity-torque inverse transformer. A gyroscope	All these transducers and transformers have equations and units of the form given in the top row. Either primary or secondary could rationally appear analogous to either an inductor or capacitor	
 Mechanical velocity-force inverse transducer. A lineal gyrator			

THE IMPEDANCE ANALOGY

Rectilineal mechanical	Rotational mechanical	Acoustic	Electric
The Conservative Ideal Inverse Transducers and Inverse Transformers, Which Couple Through Variables to Across Variables			
 Electromechanical current-force inverse transducer $E/N = N_M v$ $NI = F/N_M$ $Z = Y_M(NN_M)^2$ NN_M may be in newtons per amp or volts per m/sec	 Electrorotational current-torque inverse transducer $E/N = N_R v_R$ $NI = F_R/N_R$ $Z = Y_R(NN_R)^2$ NN_R may be in newton m per amp or volts per radian/sec	 Electroacoustic current-sound pressure inverse transducer $E/N = N_A U$ $NI = p/N_A$ $Z = Y_A(NN_A)^2$ NN_A may be in newtons/m² per amp or volts per m³/sec	 Electric current-voltage inverse transformer. A gyrator $E_1/N_1 = N_2 I_2$ $N_1 I_1 = E_2/N_2$ $Z_1 = Y_2(N_1 N_2)^2$ $N_1 N_2$ is the transfer impedance in ohms, or volts/amp
 Acoustomechanical volume velocity-force inverse transducer	 Acoustorotational volume velocity-torque inverse transducer	 Acoustic volume velocity-sound pressure inverse transformer. An acoustic gyrator	
 Rotato-rectilineal angular velocity-force inverse transducer	 Rotational angular velocity-torque inverse transducer. A gyroscope	All these transducers and transformers have equations and units of the form given in the top row. Either primary or secondary could rationally appear analogous to either an inductor or capacitor	
 Mechanical velocity-force inverse transformer. A lineal gyrator			

THE MOBILITY AND IMPEDANCE ANALOGIES

Rectilineal mechanical	Rotational mechanical	Acoustic	Electric
The Conservative Ideal Inverse Autotransformers for Connecting a Rod Diagram to a Tubing Diagram at a Change of Analogy. Through Variables Are Coupled to Across Variables			
			
$\frac{v \text{ across}}{n_M} = N_M v \text{ thru}$	$\frac{v_R \text{ across}}{n_R} = N_R v_R \text{thru}$	$\frac{U \text{ across}}{n_A} = N_A U \text{thru}$	$\frac{E_1 \text{ across}}{N_1} = N_2 I_2 \text{thru}$
$n_M F \text{ thru} = \frac{F \text{ across}}{N_M}$	$n_R F_R \text{ thru} = \frac{F_R \text{ across}}{N_R}$	$n_A p \text{ thru} = \frac{p \text{ across}}{N_A}$	$N_1 I_1 \text{ thru} = \frac{E_2 \text{ across}}{N_2}$
$z = Y_M(n_M N_M)^2$ $n_M N_M$ is the transformer factor, reciprocal of piston area, often unity	$z_R = Y_R(n_R N_R)^2$ $n_R N_R$ is the transformer factor, often unity	$z_A = Y_A(n_A N_A)^2$ $n_A N_A$ is the transformer factor, often unity	$Z_1 = Y_2(N_1 N_2)^2$ $N_1 N_2$ is the transfer impedance in ohms, or volts/amp.
Typical structure: 	Typical structure: 	Typical structure: 	Typical structure: A moving coil transducer connected mechanically to an electrostatic transducer

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Bloch, A.: Electromechanical Analogies and Their Use for the Analysis of Mechanical and Electromechanical Systems, *J. Inst. Elec. Engrs.* (*London*) **92**, 157–169 (1945, pt. I). Unfortunately, calls the mobility analogy the *inverse analogy.*

Bloch, A.: On Methods for the Construction of Networks Dual to Non-planar Networks, *Proc. Phys. Soc.* (*London*) **58**, 677–694 (1946).

Bauer, Benjamin B.: Transformer Couplings for Equivalent Network Synthesis, *J. Acoust. Soc. Am.* **25**, 837–840 (1953). Connects each element through a transformer, then gets rid of the transformers.

Trent, Horace M.: Isomorphisms between Oriented Linear Graphs and Lumped Physical Systems, *J. Acoust. Soc. Am.* **27**, 500–527 (1955). Draws mobility schematic by setting up the networks of meters required to measure F and v, etc.

Le Corbeiller, P., and Ying-Wa Yueng: Duality in Mechanics, *J. Acoust. Soc. Am.* **24**, 643–648 (November, 1950). Recommends the mobility analogy.

Firestone, Floyd A.: The Mobility Method of Computing the Vibration of Linear Mechanical and Acoustical Systems: Mechanical and Electrical Analogies, *J. Appl. Phys.* **9**, 373–387 (1938).

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Hähnle, Walter: Die Darstellung elektromechanischer Gebilde durch rein elektrische Schaltbilder, *Wissen. Veröff. Siemens-Konzern* **XI**, 1–23 (1932). Expounds both analogies.

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Circuit Theory. See Sec. 5b-14 (pp. 5-79 to 5-85) of this book.

[*Text continued from page 3-143.*]

parallel combination of these elements would with perfect analogy be the sum of the individual mechanical admittances: $Y_M = Y_{M1} + Y_{M2} + Y_{M3}$.

3m-10. Couplers: Transformers and Transducers. A coupler introduces a constraint between circuits or between different portions of the same circuit. It specifies relationships between variables at different places, a more complicated type of relationship than the 1 to 1 specified by connectors, which might be considered as simplified couplers. Ideal passive couplers transmit energy but store none. Direct couplers couple through variables to through variables and across to across, while inverse couplers couple through variables to across variables. Transformers and transducers couple like and unlike system types, respectively. In general, a transformer consists of a subtracter operated by the difference between the values of a variable at two points, actuating a multiplier, the multiplied value then being impressed as a difference between the values of a variable at two other points.

3m-11. Units. While any consistent system of units may be used in computations with analogies, the mks system is particularly advantageous since the watt will then be the unit of power in both the acoustic, mechanical, and electric portions of a transducer. Thus velocity v is in m/sec, force F in newtons, mobility z in m/sec per newton; angular velocity v_R is in radians/sec, torque F_R in newton m, rotational mobility z_R in radians/sec per newton m; volume velocity U is in m^3/sec, sound pressure p in newtons/m^2, acoustic mobility z_A in m^3/sec per newton/m^2; voltage E is in volts, current I in amperes, impedance Z in ohms. Power P is in watts and energy W in joules, throughout.

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